Aalto University
School of Science
Degree Programme in Mathematics and Systems Analysis

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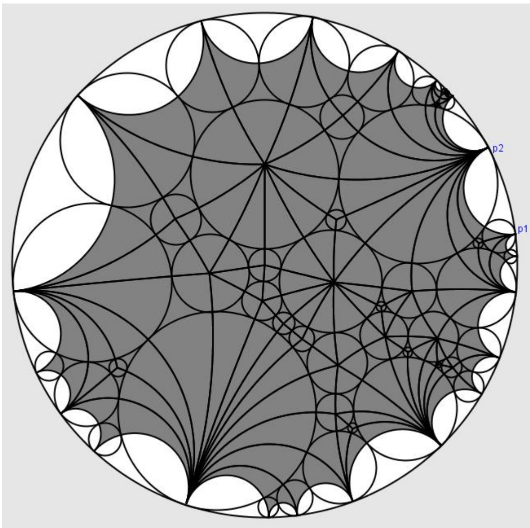
Triangulations of the topological closed disk and circle packings



Master's Thesis
Espoo, May 31, 2016

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ABSTRACT OF MASTER'S THESIS

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The main studies of this thesis are triangulations of the topological closed disk and circle packings as providers of embeddings in the hyperbolic disk for such triangulations.

Triangulations are first introduced for a more general class of topological surfaces, before focusing on triangulations of the closed disk. The combinatorial nature of triangulations is revealed and it is used to identify triangulations. Construction of bijections between sets of triangulations leads to a recursive formula for the number of rooted triangulations with given number of boundary and interior vertices. After writing the recursion in terms of generating functions, an explicit formula for the number of rooted triangulations is derived. The methods used to derive the recursive formula are also used to uniform sampling of rooted triangulations.

Circle packings are introduced at first in more general context, before concentrating on circle packings in the hyperbolic disk. The main result is that for every triangulation of the topological closed disk, there exists the maximal circle packing in the hyperbolic disk obeying the combinatorics of the triangulation. This maximal circle packing provides us with an embedding of the triangulation in the disk. These embeddings are used to visualize a collection of uniform random rooted triangulations.

In the final chapter, a definition for uniform probability measures on classes of rooted triangulations with fixed number of vertices is provided. After that, random boundary length variables from the classes to natural numbers is defined and proved that the random boundary length converges in distribution to a non-degenerate random variable, as the number of vertices tends to infinity. Respectively, after defining probability measures on classes of rooted triangulations with fixed boundary length, it is shown that an appropriately renormalized random number of vertices converges in distribution to a non-degenerate random variable, as the boundary length tends to infinity.

Keywords:	triangulations, 2-complexes, generating functions, random triangulations, circle packings, maximal packings, hyperbolic disk, embeddings, convergence in distribution
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Tämän diplomityön pääaiheina ovat topologisen suljetun kiekon triangulaatiot ja ympyräpakkaukset, jotka tarjoavat kyseisille triangulaatioille upotuksen hyperboliseen kiekkoon.
Triangulaatiot esitellään aluksi yleisemmin topologisille pinnoille, ennen keskittymistä suljetun kiekon triangulaatioihin. Triangulaatioiden kombinatorista luonnetta käytetään niiden identifioimiseksi. Bijektioiden muodostaminen triangulaatiojoukkojen välille johtaa rekursiiviseen kaavaan juurellisten triangulaatioiden lukumäärälle reuna -ja sisäpisteiden suhteen. Kun rekursio on kirjoitettu generoivien funktioiden avulle, ratkaistaan eksplisiittinen kaava juurellisten triangulaatioiden lukumäärille. Rekursiivisen kaavan muodostamisessa käytettyjä metodeita sovelletaan myös juurellisten triangulaatioiden tasaiseen otantaan.
Ympyräpakkaukset määritellään aluksi yleisemmässä kontekstissa, ennen keskittymistä hyperbolisen kiekon ympyräpakkauksiin. Päätulos on, että jokaiselle topologisen suljetun kiekon triangulaatiolle on olemassa maksimaalinen ympyräpakkaus hyperbolisessa kiekossa noudottaen kyseisen triangulaation kombinatoriikkaa. Nämä maksimaaliset ympyräpakkaukset tarjoavat suljetun kiekon triangulaatioille upotukset hyperboliseen kiekkoon. Kyseisiä upotuksia käytetään tässä työssä visualisoimaan tasaisesti valittuja satunnaisia triangulaatioita.
Viimeisessä kappaleessa luodaan tasaiset todennäköisyysmitat juurellisten triangulaatioiden luokille, joidenka triangulaatioiden pisteiden kokonaislukumäärä on kiinnitetty. Tämän jälkeen luokille määritellään reunan-pituussatunnaismuuttujat ja todistetaan, että nämä satunnaismuuttujat suppenevat jakaumaltaan, kun pisteiden kokonaislukumäärä lähestyy ääretöntä. Vastaavasti määritellään todennäköisyysmitat luokille, joidenka triangulaatioiden reunapisteiden lukumäärä on kiinnitetty ja näytetään, että sopivalla tavalla uudelleen normalisoidut pisteiden-kokonaislukumäärä-satunnaismuuttujat suppenevat jakaumaltaan, kun reunapisteiden lukumäärä lähestyy ääretöntä.

Asiasanat:	triangulaatiot, 2-kompleksit, generoivat funktiot, satunnaiset triangulaatiot, ympyräpakkaukset, maksimaaliset pakkaukset, hyperbolinen kiekko, upotukset, suppeneminen jakaumaltaan
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Espoo, May 31, 2016

Marko Voutilainen

Contents

Chapter 1

Introduction

The common thread of this thesis is triangulations of the topological closed disk. In addition to triangulations, we shall discuss circle packings and how triangulations can be embedded in geometrical space by means of them.

W. T. Tutte was first to derive a formula for the number of triangulations of the disk in his famous paper [14] from the sixties. The work of Tutte concerning triangulations of the disk was carried on by Brown in his paper [5] a couple of years later. The definition of a triangulation of Tutte did not allow such interior edges that would be incident with two exterior vertices. Brown called the triangulations enumerated by Tutte rooted strong triangulations in contrast to rooted triangulations that may contain interior edges between two boundary vertices. The term "rooted" suggest that the boundary of a triangulation has given a fixed orientation making the number of rooted triangulations considerably greater than the number of non-rooted triangulations. Among a couple of enumerations of other types of triangulations, Brown derived a formula for the number of rooted triangulations of the disk. Nowadays the motivation of study of random triangulations comes largely from the urge to discretize Feynman's path integrals of quantum gravitation. Triangulations are used to discretize topological surfaces with a metric and the related integral is evaluated over all possible paths in a triangulation. It is believed that there exists a continuous scaling limit of this discrete model that is of interest especially to physicist. [3]

The topic of circle packings is rather new in the point of view of mathematics, namely, it was introduced in a conjecture as late as 1985 by William Thurston and the first book [13], written by Kenneth Stephenson, concerning the subject was published in 2005. After a comprehensive introduction to circle packings, Stephenson defines in his book a theory of discrete analytic functions and proves that the objects of the discrete theory converge to the classical ones under refinements. He provides also a proof for the conjecture by Thurston that circle packings could be used to approximate conformal mappings. The conjecture was originally proven in [9] by Rodin and Sullivan. There exists several open questions related to circle packings. One of them is, do random discrete measures defined through circle packing embeddings of random triangulations converge to Liouville quantum gravity as the triangulation size tends to infinity? It is believed that this happens not only with circle packings, but also with all other sensible embeddings of triangulations.

We shall begin the thesis by introducing triangulations as decompositions of topological oriented surfaces and discussing how the combinatorics of triangulations can be presented in terms of abstract 2-complexes. We shall identify triangulations with their representative complexes and state conditions under which an abstract 2-complex is equivalent to a triangulation of topological oriented surface. In Section 2.2, where we restrict our considerations to triangulations of the closed disk, we shall define sets of rooted triangulations with fixed numbers of interior and boundary vertices. We shall also identify triangulations through isomorphisms of 2-complexes by essentially stating that two triangulations are the same if their combinatorics are the same. In Section 2.3 we derive a recursive formula for the number of isomorphism classes of rooted triangulations of the closed disk in the numbers of interior and boundary vertices. This is done by constructing functions that map sets of triangulations bijectively to sets of triangulations, where the triangulations on the image side have at most the same number of vertices as the triangulations of the domain. At the end of the second chapter we shall give the basic idea of generating uniform samples of rooted triangulations based on the recursive construction of Section 2.3.
The third chapter concerns the derivation of an explicit formula for the number $D_{n,m}$ of rooted triangulations with fixed numbers n and m of interior and boundary vertices. First we shall define generating functions for $D_{n,m}$ as formal power series and write the recursive formula 2.11 as a quadric equation of these power series. After extending Lagrange's inversion theorem to cover formal power series, we shall derive a solution for the equation providing us with the desired explicit formula for $D_{n,m}$.
The thread of the fourth chapter is circle packings as providers of embeddings for such 2-complexes that are equivalent to triangulations of topological oriented surfaces. Circle packings are configurations of circles in a geometric oriented surface that follow the combinatorics of the related 2-complexes. As we proceed through the chapter, we shall move from a more general setting

to a more specific one, ending up with circle packings for triangulation of the topological closed disk in the Poincaré disk, which models the 2-dimensional hyperbolic geometry. Before getting that far, we shall answer partly the question: "With a given 2-complex and set of real numbers that qualify as radii in Euclidean, spherical or hyperbolic geometry, when does there exists a circle packing in the corresponding geometrical space having the real numbers as radii and obeying the combinatorics described in the 2-complex?" However, the main theorem of the chapter is provided in Section 4.2 stating that for every triangulation of the disk there exists a so-called maximal packing in the Poincaré disk. Triangulations of the disk can be embedded in the hyperbolic disk through these circle packings and hence they are used to illustrate a collection of uniform random rooted triangulations to tie Chapters 2 and 4 tightly together.
The implementation of the uniform sampling of rooted triangulations was carried through with Matlab. The minus sign in the recursive formula 2.11 reflects the problems encountered in the implementation. A rejection process had to be included to the code slowing down the expected running time of the program exponentially as the number of interior vertices grew. Within the limits of the used mediocre computational power, this set some boundaries related to the size of a triangulation to be sampled. Anyway, a few samples of uniform random rooted triangulations is presented at the end of the fourth chapter.
The last chapter is about distributions of random rooted triangulations of the disk. We shall define probability measures on the classes of triangulations with a fixed number of vertices and on the classes of triangulations with a fixed boundary length. Since the classes with a fixed number of vertices are finite, the natural choice of the measure is the uniform one. Also the choice of probability measures on the classes with a fixed boundary length can be seen as natural, as discussed in the paper [3] by Angel and Schramm. Under the defined probability measure on a class with a fixed number of vertices, the boundary length of a triangulation is a random variable to natural numbers. We shall show that these random variables converge in distribution as the number of vertices tends to infinity. Respectively, in the case of a class with a fixed boundary length, the number of vertices of a triangulation is a random variable. However, before we can derive a similar result of convergence in distribution, we first need to use an appropriate rescaling for the random variables at hand.

Chapters 2 and 3 concerning triangulations of the disk are mainly based on Brown's paper. However, Stephenson's book [13] was relied on for the most definitions at the beginning of Chapter 2 and Munkres's book [8] was used as a reference for the concept of abstract complexes. Wilf's book was of assistance in the part of third chapter concerning formal power series. The construction of the recursion in Section 2.3 and the solving of the quadric equation in Subsection 3.3.2 follows Brown's paper with added details and illustrations.

Chapter 4 follows closely Stephenson's book [13] and the papers [6], [11] and [12]. Anderson's book [1] was used as reference for properties of the hyperbolic geometry and the Poincaré disk. The generated uniform random triangulations were embedded in the hyperbolic disk with Stephenson's CirclePack program [10].

We were not aware if there already exists derivations of the main results of the last chapter in literature, although these results might be well-known among experts in the field.

Chapter 2

Triangulations of the topological closed disk

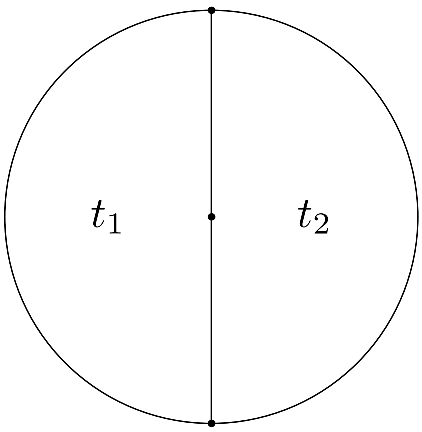
Our main focus in this chapter is on triangulations of the topological closed disk, however before proceeding to the case of the closed disk, we shall discuss triangulations in the first section of the chapter from a more general point of view. We shall initially describe triangulations in terms of topological triangles, but as we proceed we shall increasingly acknowledge the combinatorial nature of triangulations. In Section 2.3 we obtain a recursive formula for the number of isomorphism classes of rooted triangulations of the disk and in the last section of the chapter we explain how the methods of Section 2.3 can be used to construct uniform random rooted triangulations of the disk. For someone who is not familiar with concepts of topology the most efficient way of reading might be to look over briefly the first section before moving into the second section, where we restrict our considerations to triangulations of the disk. In the third section we also provide some illustrative examples of triangulations.

The construction of Section 2.3 leading to the recursive formula 2.11, which is also the main result of the chapter, follows Brown's paper [5] with some added details. Also Tutte's paper [14], where the triangulations of the disk were originally enumerated, was of assistance. The most of the definitions of the first section are based on Stephenson's book [13], but Munkres's book [8] is used as a reference for the concept of abstract complexes.

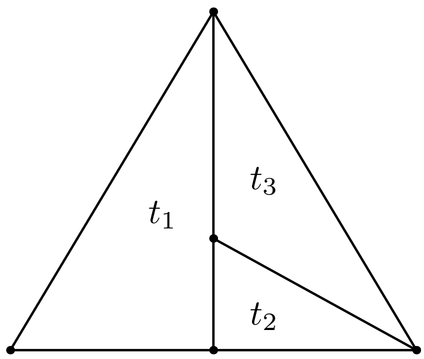
2.1 Triangulations

Definition 2.1 (Triangulations)**.** Let S be a topological surface or a topological bordered surface, that is, a connected Hausdorff space where every point has a neighbourhood, which is homeomorphic to an open subset of the Euclidean plane or to an open subset of the closed Euclidean upper half-plane (with the subspace topology) respectively. In addition, let S be oriented, that is, it has an atlas having orientation-preserving transition maps. By a triangulation T of S we shall mean a locally finite decomposition of S into a collection of topological closed triangles $T = \{t_i\}$, where any two triangles are either disjoint, intersect in a single complete edge or intersect in a single vertex.

In the definition above, locally finiteness means that every point of S has a neighbourhood, which intersects at most finitely many triangles of T. By a topological triangle we shall mean a topological closed disk (a topological space, which is homeomorphic to the closed disk of the Euclidean plane) with three marked points on its boundary. Two examples of decompositions into topological closed triangles that do not qualify as triangulations are provided in Figure 2.1. In Subfigure 2.1a the triangles t_1 and t_2 share two edges. In Subfigure 2.1b the triangle t_1 shares only a part of one of its edges with the triangle t_2. The same holds between the triangles t_1 and t_3.
The abstract relationships, combinatorics, of a triangulation T can be presented as an abstract simplicial 2-complex (2-complex for short). After we have given a definition for abstract simplicial complexes, we shall elaborate further the relationship between triangulations and 2-complexes.



(a)



(b)

Figure 2.1: Examples that do not qualify as triangulations

Definition 2.2 (k-complexes)**.** An abstract simplicial k-complex $\mathscr{L}$ is a collection of finite non-empty sets such that if $A \in \mathscr{L}$, then every non-empty subset of A belongs also to $\mathscr{L}$. In addition, $\mathscr{L}$ contains at least one set with

$k+1$ elements, but no sets with more than $k+1$ elements. The sets of $\mathscr{L}$ with $n+1$ elements are called abstract n-simplexes (n-simplexes for short).

If the combinatorics of some triangulation T of an oriented (bordered) surface is coded in a 2-complex $\mathscr{K}$, then each 2-simplex of $\mathscr{K}$ corresponds to a triangle of T. We shall call 2-simplexes of a 2-complex faces, although in the general terminology of abstract simplexes "face" can refer to any non-empty subset of a simplex. We may use the word "face" also when referring to the triangles of the triangulation T, since in the end we do not really distinguish triangulations from their complexes. The 1-simplexes of $\mathscr{K}$ correspond to the edges of T and 0-simplexes of $\mathscr{K}$ correspond to the vertices of T. The 1-simplexes and 0-simplexes of a 2-complex are called edges and vertices respectively. If $\mathscr{K}$ has a face incident with vertices v, u and w, we use notation $\langle vuw \rangle$ for the face and notation $\langle vu \rangle$ for the edge incident with the vertices v and u. We shall denote the oriented edge directed from v to u with $\langle vu \rangle^*$. By $\langle vuw \rangle^*$ we shall mean the face $\langle vuw \rangle$ with induced orientations $\langle vu \rangle^*$, $\langle uw \rangle^*$ and $\langle wv \rangle^*$ on its boundary edges.
Edges of $\mathscr{K}$ that belong only to one of the faces are called boundary edges and the other edges, belonging to two of the faces are called interior edges. Vertices that belong to two of the boundary edges are called boundary vertices and the other vertices, belonging to none of the boundary edges are called interior vertices. Faces having at least one boundary edge are called boundary faces and the others are called interior faces. The classes of boundary and interior simplexes of $\mathscr{K}$ are denoted with $\mathscr{K}_{bnd}$ and $\mathscr{K}_{int}$ respectively. The same terminology and notations are applied also to the triangulation T. We say that $\mathscr{K}$ is (simplicially) equivalent to a triangulation of an oriented topological (bordered) surface and that T is a realization of $\mathscr{K}$ as such a triangulation.

Next we shall extend the topological concepts of connectedness and simply connectedness to cover the 2-complexes. The need for understanding the property of connectedness of 2-complexes arise still in this section. Although the concept of simply connectedness is not required until in the next section (and also in Chapter 4), it feels natural to define both properties here in the same place.

Definition 2.3 (Chains)**.** Let $\mathscr{L}$ be an abstract simplicial 2-complex. A chain is a sequence $\Gamma = \{f_0, f_1, ..., f_n\}$ of faces of $\mathscr{L}$, where every face f_i shares one edge with the next face f_{i+1} of the chain. If $f_0 = f_n$, then the chain Γ is a closed chain. A subchain $\gamma = \{f_j, ..., f_k\} \subset \Gamma$ is said to be local at a vertex v, if v belongs to every face of γ. A chain Γ' is said to be a local modification of Γ, if it is obtained by replacing γ in Γ with any other local subchain γ' at v having the same first and last face as γ. We say that two chains Γ_1 and Γ_2 are homotopic if one can be obtained from the other by a finite number of local modifications.

Definition 2.4 (Connectedness)**.** Let $\mathscr{L}$ be an abstract simplicial 2-complex and let f_0 and f_1 be two faces of $\mathscr{L}$. If for every pair (f_0, f_1) of faces there exists a chain from f_0 to f_1, then $\mathscr{L}$ is connected.

Definition 2.5 (Simply connectedness)**.** Let $\mathscr{L}$ be a connected abstract simplicial 2-complex and let f_0 be an arbitrary face of $\mathscr{L}$. If every closed chain $\Gamma = \{f_0, ..., f_0\}$ is homotopic to the null chain $\Gamma_0 = \{f_0\}$, then $\mathscr{L}$ is simply connected.

Before proceeding to study the case of triangulations of the topological closed disk, we shall first state a proposition that complements our considerations about the relationship between triangulations and 2-complexes.

Proposition 2.6. *An abstract simplicial 2-complex $\mathscr{L}$ represents a triangulation of an oriented topological (bordered) surface if and only if*

(1) $\mathscr{L}$ is connected.

(2) Any two faces are either disjoint, share a single edge or share a single vertex.

(3) It is possible to give an orientation for the boundary of each of the faces in such a way that if two faces share an edge, then they induce opposite orientations on that edge.

(4) Each edge belongs to either one or two faces.

(5) Each vertex belongs either to none of the boundary edges or to two of the boundary edges.

(6) Each vertex belongs at least to one face and at most to finite number of faces. These faces can be ordered in such a way that each face shares an edge with the next one.

Notice that if an abstract 2-complex $\mathscr{K}$ satisfies the lemma above, we could define simply connectedness also by saying that $\mathscr{K}$ is simply connected if and only if the topological surface it triangulates is simply connected.

2.2 Triangulations of the topological closed disk

In this section we shall focus on the case where the surface S to be triangulated is the topological closed disk. The topological closed disk is in fact a bordered surface, since neighbourhoods of its boundary points are homeomorphic to open subsets of the Euclidean closed half-plane. To start with, we shall classify triangulations of a topological space with the topology of the closed disk by the numbers of their boundary and interior vertices. After that, we shall give triangulations an identification through isomorphisms between their 2-complexes. In the end of the section we shall justify that the choice for the representative of the topological closed disk is irrelevant. This will induct us into the next section, where we shall start the enumeration of isomorphism classes of triangulations of the closed disk.

Definition 2.7. Let Ω be a topological space with the topology of the closed disk. By a triangulation T of Ω of type $[n, m]$ we shall mean a triangulation of Ω that has n interior vertices and $m + 3$ boundary vertices. That is, n of the $n + m + 3$ vertices of T lie in the interior of Ω and $m + 3$ of the vertices lie on the boundary of Ω.

For the sake of simplicity let us consider the closed unit disk of the Euclidean plane for a moment. If triangulations would be identified in terms of their topological triangles as sets of points, the number of any type of triangulations would be uncountable infinite. The natural way to give an identification for triangulations is through their combinatorics, that is, through their representative 2-complexes. By doing so, the number of any type of triangulations (after identification) is going to be finite.

Definition 2.8 (Isomorphisms)**.** Let Ω and Ω' be two topological spaces with the topology of the closed disk. Let T be a triangulation of Ω and T' be a triangulation of Ω', and let $\mathscr{T}$ and $\mathscr{T}'$ be the respective 2-complexes of the triangulations. Then an isomorphism $\psi : \mathscr{T} \to \mathscr{T}'$ is a bijection, which maps vertices to vertices, edges to edges and faces to faces. In addition, if $A, B \in \mathscr{T}$ then $\psi(A)$ and $\psi(B)$ are incident in $\mathscr{T}'$ if and only if A and B are incident in $\mathscr{T}$. If such an isomorphism exists, we say that T is isomorphic with T' ($\mathscr{T}$ is isomorphic with $\mathscr{T}'$).

Now since we are identified triangulations in terms of their 2-complexes by stating that two triangulations are the same if their combinatorics are the same, we shall start to discuss triangulations in terms of 2-complexes. For the same reason there is no need to discuss several topological spaces with the topology of the closed disk. Therefore, from now on we shall discuss only

the closed disk. However, at the end of section we shall justify more carefully why we have the freedom of choice regarding the representative of the closed disk when it comes to the enumeration of triangulations.
The next proposition is a special case of Proposition 2.6 providing us with the relationship between triangulations of the closed disk and 2-complexes.

Proposition 2.9. *Let 2-complex $\mathscr{K}$ be a triangulation of an oriented topological (bordered) surface. That is, $\mathscr{K}$ satisfies the conditions of Proposition 2.6. If in addition $\mathscr{K}$ is finite, simply connected and has non-empty boundary, then $\mathscr{K}$ is (simplicially) equivalent to a triangulation of the closed disk.*

Next we shall define rooted triangulations together with root-isomorphisms. Our ultimate goal in this chapter is to enumerate the isomorphism classes of rooted triangulations of type $[n, m]$.

Definition 2.10 (Root-isomorphisms)**.** Let $\mathscr{T}$ be a triangulation of the closed disk and let $\langle p_1p_2\rangle$ be a boundary edge of $\mathscr{T}$. Then we shall call the ordered pair $(\mathscr{T}, \langle p_1p_2\rangle^*)$ a rooted triangulation. Let $(\mathscr{T}', \langle q_1q_2\rangle^*)$ be another rooted triangulation of the closed disk. Then a root-isomorphism is an isomorphism $\psi : \mathscr{T} \to \mathscr{T}'$ that maps p_1 to q_1 and p_2 to q_2. If such a root-isomorphism exists, we say that $(\mathscr{T}, \langle p_1p_2\rangle^*)$ is isomorphic with $(\mathscr{T}', \langle q_1q_2\rangle^*)$ and we write $(\mathscr{T}, \langle p_1p_2\rangle^*) \cong (\mathscr{T}', \langle q_1q_2\rangle^*)$.

Let us now consider two topological spaces Ω and Ω' with the topology of the closed disk and a homeomorphism $\phi : \Omega \to \Omega'$ between them. Let us assume that $(\mathscr{T}_1, \langle p_1p_2\rangle^*)$ and $(\mathscr{T}_2, \langle q_1q_2\rangle^*)$ are isomorphic rooted triangulations of Ω with a root-isomorphism $\psi : (\mathscr{T}_1, \langle p_1p_2\rangle^*) \to (\mathscr{T}_2, \langle q_1q_2\rangle^*)$ between them. The homeomorphism ϕ induces isomorphisms $\tilde{\psi}_1$ and $\tilde{\psi}_2$ that map the two rooted triangulation of Ω to the rooted triangulations

$$(\tilde{\psi}_1(\mathscr{T}_1), \langle\tilde{\psi}_1(p_1)\tilde{\psi}_1(p_2)\rangle^*) \quad \text{and} \quad (\tilde{\psi}_2(\mathscr{T}_2), \langle\tilde{\psi}_2(q_1)\tilde{\psi}_2(q_2)\rangle^*)$$

of Ω'. The two rooted triangulations of Ω' are isomorphic with the root-isomorphism

$$\tilde{\psi}_2 \circ \psi \circ \tilde{\psi}_1^{-1}$$

between them. Respectively, if the images of two rooted triangulations of Ω under $\tilde{\psi}$ are root-isomorphic in Ω', then so are the original rooted triangulations in Ω. Therefore the set of rooted triangulations of Ω is isomorphically equivalent to the set of rooted triangulations of Ω'. When we start the enumeration of isomorphism classes of rooted triangulations in the next section, this observation guarantees that we have the freedom of choice when selecting our representative for the topological closed disk that we are triangulating. Two natural choices are a closed disk of the Euclidean plane and a polygon, whose number of vertices corresponds to the number of boundary vertices of the triangulation at hand.

2.3 Recursion for the number of triangulations

In this section our goal is to obtain a recursive formula for the number of isomorphism classes of rooted triangulations of the closed disk. We shall denote the set of rooted triangulations of type $[n, m]$ with $\mathcal{T}_{n,m}$ and the number of elements (after identifications through isomorphisms) in $\mathcal{T}_{n,m}$ with $D_{n,m}$. From now on we shall only consider rooted triangulations of the closed disk and therefore in the future we shall always refer with "triangulation" to a rooted triangulation of the closed disk.

Let $(\mathscr{T}, \langle p_1p_2\rangle^*) \in \mathcal{T}_{n,m}$ with $n + m \neq 0$ be arbitrary. Let $\langle ap_1p_2\rangle$ be the face incident with the root edge $\langle p_1p_2\rangle$ and $\mathscr{L}$ be the abstract 2-complex obtained by erasing $\langle p_1p_2\rangle$ and $\langle ap_1p_2\rangle$ from $\mathscr{T}$. Let us consider the case of a being a boundary vertex and the case of a being an interior vertex separately.

Case 1: $a \in \mathscr{T}_{bnd}$

Assume that $\langle ap_i\rangle \notin \mathscr{T}_{bnd}$ for $i \in \{1, 2\}$. Then by duplicating the vertex a, the complex $\mathscr{L}$ can be divided into two connected pieces $\mathscr{T}_1$ and $\mathscr{T}_2$ both having the vertex a as an element. By choosing $\langle ap_1\rangle^*$ and $\langle ap_2\rangle^*$ as the roots of $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively, we obtain new triangulations $(\mathscr{T}_1, \langle ap_1\rangle^*)$ and $(\mathscr{T}_2, \langle ap_2\rangle^*)$ of type $[n_1, m_1]$ and $[n_2, m_2]$ respectively.
Assume that $\langle ap_i\rangle \in \mathscr{T}_{bnd}$ for $i = 1$ or for $i = 2$. The restriction $n + m \neq 0$ guarantees that both edges can not be boundary edges. Let us first assume that $\langle ap_1\rangle \in \mathscr{T}_{bnd}$. Now by repeating the construction used above we will end up with a triangulation $(\mathscr{T}_2, \langle ap_2\rangle^*)$ of type $[n_2, m_2]$ and an oriented edge $\langle ap_1\rangle^*$. Similarly, if $\langle ap_2\rangle \in \mathscr{T}_{bnd}$, then we will end up with a triangulation $(\mathscr{T}_1, \langle ap_1\rangle^*)$ of type $[n_1, m_1]$ and an oriented edge $\langle ap_2\rangle^*$. We shall classify oriented edges as triangulations of type $[0, -1]$ and use the same notation for them as for the actual triangulations.
Regardless if one of the edges $\langle ap_i\rangle$ for $i \in \{1, 2\}$ is an external edge or not, the pair of triangulations resulting from the construction satisfies

$$m_1+3+m_2+3 = m+3+1 \iff m_1+m_2 = m-2, \quad \text{and} \quad n_1+n_2 = n. \tag{2.1}$$

Let us denote the set of triangulations of type $[n, m]$ having a as a boundary vertex with $\tilde{\mathcal{T}}_{n,m}$. Our goal is to show that the method used above to construct from $(\mathscr{T}, \langle p_1p_2\rangle^*) \in \tilde{\mathcal{T}}_{n,m}$ a pair $((\mathscr{T}_1, \langle ap_1\rangle^*), (\mathscr{T}_2, \langle ap_2\rangle^*))$ of new triangulations defines a bijection

$$\tilde{g} : \tilde{\mathcal{T}}_{n,m} \to \bigcup_{\substack{m_1+m_2=m-2\\ n_1+n_2=n\\ m_i,n_i\geq 0}} (\mathcal{T}_{n_1,m_1} \times \mathcal{T}_{n_2,m_2}) \cup (\mathcal{T}_{0,-1} \times \mathcal{T}_{n,m-1}) \cup (\mathcal{T}_{n,m-1} \times \mathcal{T}_{0,-1}). \quad (2.2)$$

Since $n + m \neq 0$, the codomain of $\tilde{g}$ is a disjoint union. The sets $\mathcal{T}_{n,-1}$ with $n \geq 1$ are considered here as empty sets.
We shall choose so that the starting vertex p_1 of the root edge $\langle p_1p_2\rangle^*$ of $\mathscr{T}$ is mapped by $\tilde{g}$ always on the first component of the Cartesian products of the image side. Respectively, the end vertex p_2 of the root edge will be then mapped on the second component of the Cartesian products on the image side. Naturally, the opposite choice could be done as well. As long as we make the choice consistently, $\tilde{g}$ is a well-defined function. The function $\tilde{g}$ is illustrated in Figure 2.2 through topological realizations of two triangulations.

Let us next define a candidate $\tilde{g}^{-1}$ for the inverse of $\tilde{g}$. Let us assume that $((\mathscr{T}_1', \langle ap_1\rangle^*), (\mathscr{T}_2', \langle ap_2\rangle^*))$ is an arbitrary pair of rooted triangulations, where $\mathscr{T}_1'$ is of type $[n_1, m_1]$ and $\mathscr{T}_2'$ is of type $[n_2, m_2]$ satisfying the conditions of Equation 2.1 (allowing the case where $m_i = -1$ for either $i = 1$ or $i = 2$). Then $\tilde{g}^{-1}$ shall join $\mathscr{T}_1'$ together with $\mathscr{T}_2'$ by identifying the starting vertices of the roots of the triangulations. In addition, $\tilde{g}^{-1}$ shall add the edge $\langle p_1p_2\rangle$ and the face $\langle ap_1p_2\rangle$ to the amalgamation of $\mathscr{T}_1'$ and $\mathscr{T}_2'$. Let us denote the resulting non-rooted triangulation with $\mathscr{T}'$. Finally, $\tilde{g}^{-1}$ pairs $\mathscr{T}'$ up with the root $\langle p_1p_2\rangle^*$ resulting a triangulation $(\mathscr{T}', \langle p_1p_2\rangle^*) \in \tilde{\mathcal{T}}_{n,m}$. If the choice of the root is made consistently, then $\tilde{g}^{-1}$ is a well-defined function. The function $\tilde{g}^{-1}$ is illustrated in Figure 2.3 through topological realizations of a pair of triangulations.

By the definitions of the two functions it is easy to see that

$$\tilde{g}^{-1}\left(\tilde{g}((\mathscr{T}, \langle p_1p_2\rangle^*))\right) = (\mathscr{T}, \langle p_1p_2\rangle^*)$$

and

$$\tilde{g}\left(\tilde{g}^{-1}\left(((\mathscr{T}_1', \langle ap_1\rangle^*), (\mathscr{T}_2', \langle ap_2\rangle^*))\right)\right) = (\mathscr{T}_1', \langle ap_1\rangle^*), (\mathscr{T}_2', \langle ap_2\rangle^*)).$$

Since $(\mathscr{T}, \langle p_1p_2\rangle^*)$ and $((\mathscr{T}_1', \langle ap_1\rangle^*), (\mathscr{T}_2', \langle ap_2\rangle^*))$ were arbitrary, $\tilde{g}$ is a bijection. Hence the cardinalities of the domain and the codomain of $\tilde{g}$ are equal.

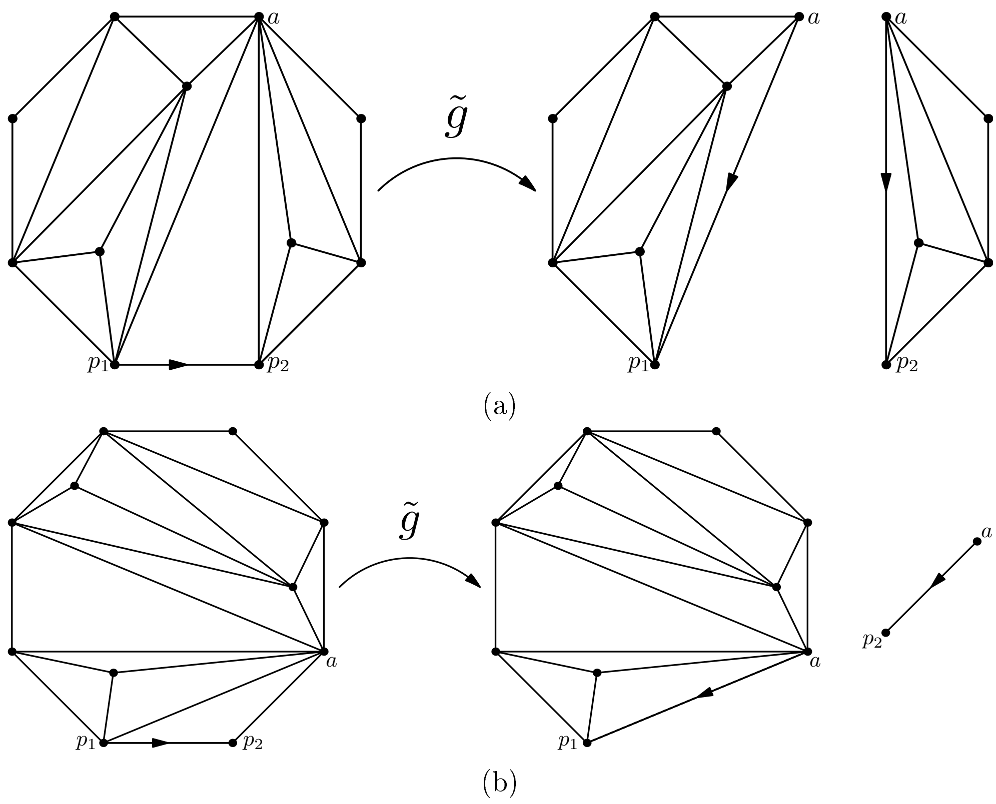


Figure 2.2: Example triangulations mapped by the function $\tilde{g}$.

By remembering that the codomain in Equation 2.2 is a disjoint union and that $D_{0,-1} = 1$, we obtain the following formula for the number $\tilde{D}_{n,m}$ of rooted triangulations of type $[n, m]$ having a as a boundary vertex

$$\tilde{D}_{n,m} = 2D_{n,m-1} + \sum_{\substack{m_1+m_2=m-2 \\ n_1+n_2=n \\ m_i,n_i \geq 0}} D_{n_1,m_1} D_{n_2,m_2}, \quad n + m \neq 0. \tag{2.3}$$

Case 2: $a \in \mathscr{T}_{int}$

Notice that Case 2 can occur only if $n \geq 1$.
In this case, the 2-complex $\mathscr{L}$ is already a non-rooted triangulation. By choosing $\langle ap_1 \rangle^*$ as the root for $\mathscr{L}$, we obtain a triangulation $(\mathscr{L}, \langle ap_1 \rangle^*)$ of type $[n-1, m+1]$. Let us denote the set of triangulations of type $[n, m]$ having a as an interior vertex with $\hat{\mathcal{T}}_{n,m}$. The used construction defines a function

$$\hat{g} : \hat{\mathcal{T}}_{n,m} \to \mathcal{T}_{n-1,m+1} \quad \text{for } n \geq 1,$$

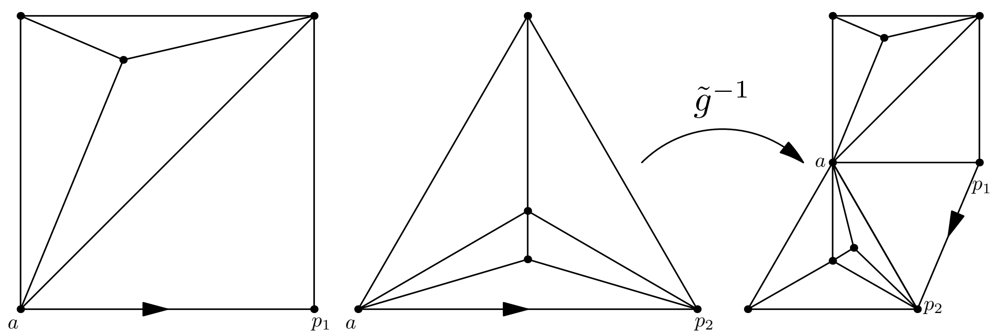


Figure 2.3: An example pair of triangulations mapped by the function $\tilde{g}^{-1}$.

but it is not a bijection. Consider a triangulation $(\mathscr{T}_1, \langle ap_1\rangle^*)$ of type $[n-1, m+1]$ with $n \geq 1$ and $m \geq 0$. Let us denote the other boundary vertex (apart from p_1) incident with a through a boundary edge with p_2. If $\mathscr{T}_1$ has the edge $\langle p_1p_2\rangle$ as an element, then $\mathscr{T}_1$ can not be the image of $\mathscr{T}$ under $\hat{g}$. Let us denote the set of triangulations of type $[n-1, m+1]$ containing the edge $\langle p_1p_2\rangle$ with $\mathcal{T}^*_{n-1,m+1}$. We shall show that our construction defines a bijection

$$\hat{g} : \hat{\mathcal{T}}_{n,m} \to \mathcal{T}_{n-1,m+1} \setminus \mathcal{T}^*_{n-1,m+1} \quad \text{for } n \geq 1. \tag{2.4}$$

Let us define a candidate $\hat{g}^{-1}$ for the inverse of $\hat{g}$. Assume that

$$(\mathscr{T}_1, \langle ap_1\rangle^*) \in \mathcal{T}_{n-1,m+1} \setminus \mathcal{T}^*_{n-1,m+1}$$

is arbitrary. Then $\hat{g}^{-1}$ shall add the edge $\langle p_1p_2\rangle$ and the triangle $\langle ap_1p_2\rangle$ to $\mathscr{T}_1$. The resulting non-rooted triangulation $\mathscr{T}'$ is of type $[n, m]$. Finally, $\hat{g}^{-1}$ pairs $\mathscr{T}'$ up with the root $\langle p_1p_2\rangle^*$ resulting a triangulation $(\mathscr{T}', \langle p_1p_2\rangle^*) \in \hat{\mathcal{T}}_{n,m}$. The functions $\hat{g}$ and $\hat{g}^{-1}$ are illustrated in Figure 2.4 through topological representatives of two triangulations.

By the definitions of the two functions

$$\hat{g}^{-1}\left(\hat{g}((\mathscr{T}, \langle p_1p_2\rangle^*))\right) = (\mathscr{T}, \langle p_1p_2\rangle^*)$$

and

$$\hat{g}\left(\hat{g}^{-1}((\mathscr{T}_1, \langle ap_1\rangle^*))\right) = (\mathscr{T}_1, \langle ap_1\rangle^*).$$

Since the two triangulations were arbitrary, $\hat{g}$ is a bijection. Let us denote the number of the elements in the set $\mathcal{T}^*_{n-1,m+1}$ with $D^*_{n-1,m+1}$. To be able to obtain a recursive formula for the number $D_{n,m}$ we still need to express $D^*_{n-1,m+1}$ in other terms.

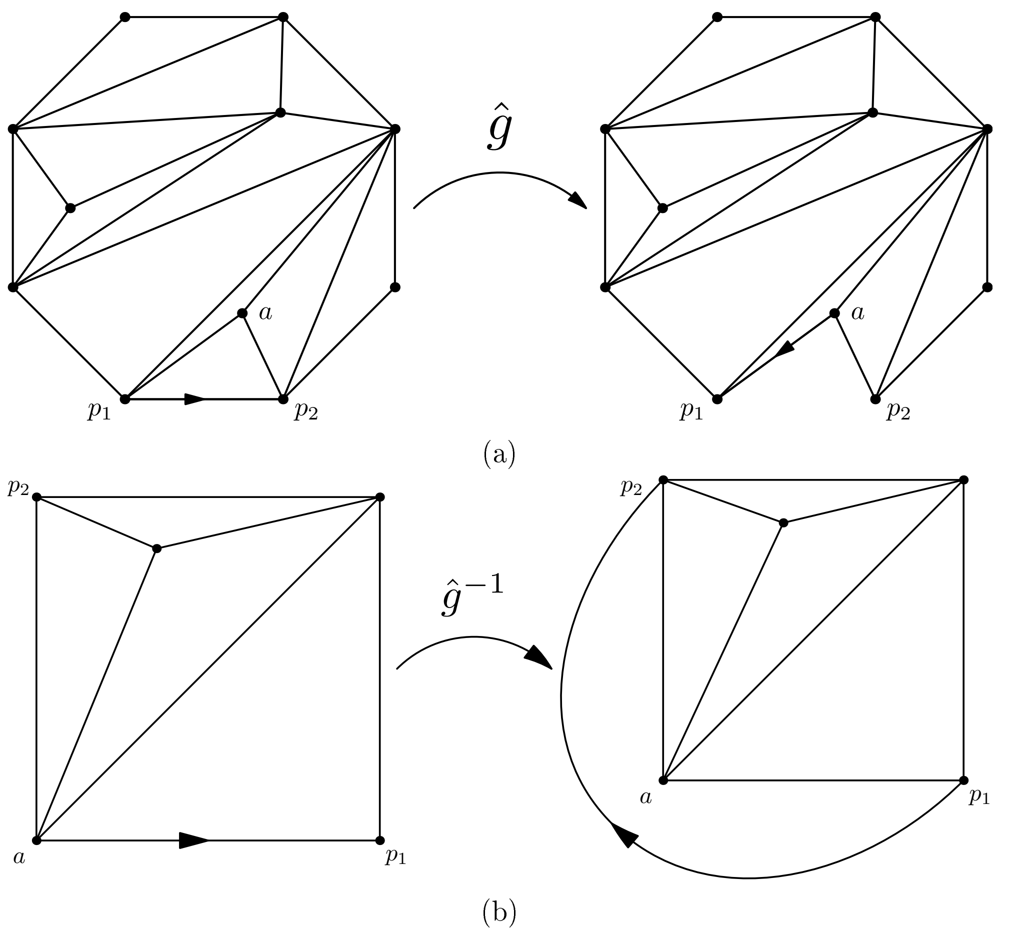


Figure 2.4: Example triangulations mapped by the functions $\hat{g}$ and $\hat{g}^{-1}$.

Let us assume that $(\mathscr{T}^*, \langle ap_1\rangle^*) \in \mathcal{T}^*_{n-1,m+1}$. Then by using the same notation as earlier, $\langle p_1p_2\rangle$ is an interior edge of $\mathscr{T}^*$. By duplicating the edge $\langle p_1p_2\rangle$ and the vertices p_1 and p_2 we may divide $\mathscr{T}^*$ along $\langle p_1p_2\rangle$ into two pieces $\mathscr{T}_1$ and $\mathscr{T}_2$ both having the edge and the two vertices as elements. $\mathscr{T}_1$ and $\mathscr{T}_2$ are non-rooted triangulations of type $[n_1, 0]$ and $[n_2, m]$ respectively that satisfy $n_1 + n_2 = n - 1$. By pairing them both up with the root $\langle p_1p_2\rangle^*$, we obtain two triangulations of the same types. This construction defines yet another bijective function

$$\phi : \mathcal{T}^*_{n-1,m+1} \to \bigcup_{n_1+n_2=n-1} (\mathcal{T}_{n_1,0} \times \mathcal{T}_{n_2,m}) \,. \tag{2.5}$$

Let us construct the inverse function ϕ^{-1} of ϕ. Assume that

$$((\mathscr{T}_1', \langle p_1p_2\rangle^*), (\mathscr{T}_2', \langle p_1p_2\rangle^*)) \in \mathcal{T}_{n_1,0} \times \mathcal{T}_{n_2,m}$$

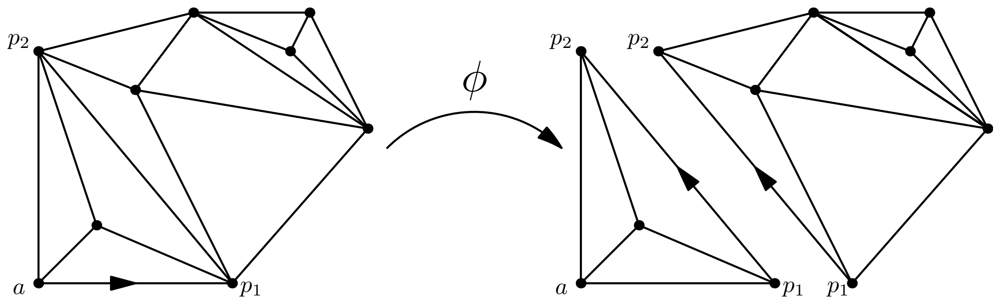


Figure 2.5: An example triangulation mapped by the function ϕ.

with $n_1 + n_2 = n - 1$. Let us denote the third boundary vertex (apart from p_1 and p_2) of $\mathscr{T}_1'$ with a. The function ϕ^{-1} shall join $\mathscr{T}_1'$ and $\mathscr{T}_2'$ together by identifying the root edges and the vertices incident with the root edges (with the notation we are using this corresponds to taking union of the two abstract 2-complexes). Pairing this amalgamation up with the root $\langle ap_1\rangle^*$ results a triangulation of type $[n-1, m+1]$. This procedure defines ϕ^{-1} and similarly as earlier it can be checked that ϕ^{-1} is the inverse of ϕ. The function ϕ is illustrated in Figure 2.5 through a topological representative of a triangulation.

By Equations 2.4 and 2.5 we know that the number $\hat{D}_{n,m}$ of rooted triangulations of type $[n, m]$ having a as an interior vertex satisfies

$$\begin{aligned}\hat{D}_{n,m} &= D_{n-1,m+1} - D^*_{n-1,m+1} \\ &= D_{n-1,m+1} - \sum_{n_1+n_2=n-1} D_{n_1,0} D_{n_2,m} \quad \text{when } n \geq 1.\end{aligned} \tag{2.6}$$

Theorem 2.11. *For the number $D_{n,m}$ of rooted triangulations of type $[n, m]$ holds*

$$D_{n,m} = 2D_{n,m-1} + D_{n-1,m+1} + \sum_{\substack{m_1+m_2=m-2 \\ n_1+n_2=n}} D_{n_1,m_1} D_{n_2,m_2} - \sum_{n_1+n_2=n-1} D_{n_1,0} D_{n_2,m} \tag{2.7}$$

when $n + m \geq 1$. In addition, $D_{0,0} = 1$.

Proof. The recursive formula of the theorem is formed by combining Equations 2.3 and 2.6. If all the numbers having a negative index are interpreted as zeros, then the formula will hold for every non-negative n and m, such that $n + m \neq 0$. The set $\mathcal{T}_{0,0}$ consists of the trivial triangulation and hence $D_{0,0} = 1$. □

Let us assume that we want to find out the numeric value of $D_{n,m}$. From Equation 2.7 can be seen that if we know all the numbers $D_{n',m'}$ with $n' \leq n$ and $m' \leq m-1$, and all the numbers D_{n^*,m^*} with $n^* \leq n-1$ and $m^* \leq m+1$, then $D_{n,m}$ is solvable. In fact it is sufficient, but not necessary to know all these numbers explicitly.
To give a more explicit example, let us assume that we wish to know the numeric value of $D_{2,m}$. From $D_{0,0}$ it is possible to determine recursively the numeric values of $D_{0,1}, D_{0,2}, ..., D_{0,m+2}$. After that, it is possible to determine $D_{1,0}, ..., D_{1,m+1}$ and finally we can find out the numeric values of $D_{2,0}, D_{2,1}, ..., D_{2,m}$.

2.4 Generation of uniform random triangulations

The method used in the previous section to define the recursive formula for the number of triangulations can be applied to generate random triangulations of the closed disk. In this section we shall explain the general idea of choosing a triangulation $(\mathscr{T}, \langle p_1 p_2 \rangle^*)$ uniformly from the set $\mathcal{T}_{n,m}$. Examples of uniformly chosen triangulations are provided in Section 4.3, where they are embedded in the hyperbolic disk through their so-called maximal circle packings.

Let us assume that we have randomly and uniformly picked a triangulation $(\mathscr{T}, \langle p_1 p_2 \rangle^*)$ from $\mathcal{T}_{n,m}$, but we do not know how does it look like. What we do know is that it has $m+3$ boundary edges and vertices. First we wish to determine how does the face $\langle a p_1 p_2 \rangle$ incident with the root edge look like. In general, there are four different outcomes.

Case 1: The vertex a is an interior vertex.

Case 2: The vertex a is a boundary vertex, and the edges $\langle a p_1 \rangle$ and $\langle a p_2 \rangle$ are interior edges.

Case 3: The vertex a is a boundary vertex, the edge $\langle a p_1 \rangle$ is an interior edge and the edge $\langle a p_2 \rangle$ is a boundary edge.

Case 4: The vertex a is a boundary vertex, the edge $\langle a p_1 \rangle$ is a boundary edge and the edge $\langle a p_2 \rangle$ is an interior edge.

We assume here that $\mathscr{T}$ is not the trivial triangulation, that is, $n+m \neq 0$ and therefore the case where both of the edges $\langle a p_1 \rangle$ and $\langle a p_2 \rangle$ would be

boundary edges is not possible.
By the bijections $\tilde{g}$, $\hat{g}$ and ϕ (see Equations 2.2, 2.4 and 2.5) of the previous section, we know for example that the probability for Case 1 to occur is

$$\frac{D_{n-1,m+1} - \sum\limits_{n_1+n_2=n-1} D_{n_1,0} D_{n_2,m}}{D_{n,m}}$$

and the probability for Cases 3 and 4 to occur is $D_{n,m-1}/D_{n,m}$. If Case 3 or Case 4 occurs, then the removal of the edge $\langle p_1 p_2 \rangle$ and the triangle $\langle a p_1 p_2 \rangle$ (we shall call this procedure the recursion step) leaves us with an oriented edge and an uniform random triangulation of type $[n, m-1]$. If Case 2 occurs, the recursion step gives us two independent uniform random triangulations that satisfy the conditions in Equation 2.1. If Case 1 occurs, the recursion step leaves us with a random triangulation, which is chosen uniformly from the subset $\mathcal{T}_{n-1,m+1} \setminus \mathcal{T}^*_{n-1,m+1}$ of $\mathcal{T}_{n-1,m+1}$. For further details, see the previous section.
We can continue the recursion in the new triangulation(s) until we have only deterministic triangulations in our hands. The deterministic triangulations are the trivial triangulation, the oriented edge and the triangulation of type $[1, 0]$. Since all steps of this recursive construction are bijective, the set of the deterministic triangles in the end and the recursion paths (the occurred cases of recursion) leading to each one of them define an unique triangulation of type $[n, m]$.
Case 1 is a bit problematic in practice. When uniform random triangulations were generated, the uniform sampling of triangulations from the set $\mathcal{T}_{n-1,m+1} \setminus \mathcal{T}^*_{n-1,m+1}$ was implemented by uniformly sampling triangulations from the set $\mathcal{T}_{n-1,m+1}$ and by discarding triangulations that had the forbidden edge. Since Case 1 is the only case of the recursion that brings down the number of interior vertices, the method of rejection slows down the generation process exponentially as the number of interior vertices grows. Nevertheless, uniform random triangulations with desired degree of inner structure were generated in reasonable time and they are visualized in Section 4.3.

Chapter 3

The number of triangulations

In this chapter we shall continue to work with the recursive formula 2.7 of Chapter 2. Although the formula determines uniquely the number $D_{n,m}$, the method of recursion becomes increasingly time consuming as the number of vertices grows. Therefore we wish to discover an explicit formula for the number of triangulations. In order to do that we shall first define generating functions for $D_{n,m}$, that is, formal power series having the numbers $D_{n,m}$ as coefficients of their terms. Then in the second section we shall write the recursive formula 2.7 as a quadric equation of one of these generating functions. In the beginning of the third section we shall shortly introduce formal power series and Lagrange's inversion theorem, which is applied later in the section as we search for an explicit formula for the number $D_{n,m}$ of triangulations of type $[n, m]$.
The chapter follows closely Brown's paper [5] with some added details, but also Tutte's paper [14] was of assistance. Wilf's book [15] is relied on for the part that concerns properties of formal power series and for the proof of Lagrange's theorem.

3.1 Generating functions

Let us give an enumeration for $D_{n,m}$ through the following generating functions

$$
\begin{aligned}
D_{.m}(x) &= \sum_{n=0}^{\infty} D_{n,m} x^n \\
D(x,y) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} D_{n,m} x^n y^m \\
&= \sum_{m=0}^{\infty} D_{.m}(x) y^m
\end{aligned} \tag{3.1}
$$

The last one of the functions can be decomposed into two parts as

$$
\begin{aligned}
D(x,y) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\tilde{D}_{n,m} + \hat{D}_{n,m}) x^n y^m \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \tilde{D}_{n,m} x^n y^m + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \hat{D}_{n,m} x^n y^m \\
&=: \tilde{D}(x,y) + \hat{D}(x,y),
\end{aligned} \tag{3.2}
$$

where $\tilde{D}_{n,m}$ is the number of triangulations of type $[n,m]$ having the vertex incident with the root edge as a boundary vertex and $\hat{D}_{n,m}$ is the number of triangulations of type $[n,m]$ having the vertex incident with the root edge as an interior vertex.
The generating functions above will be considered as formal power series. That is, our focus of interest will rather be on the coefficients than on the convergence of the series.

We shall use the general notation for the coefficient extraction operator $[*]$ applied to a formal power series.

Example 3.1. Assume that $f(x)$ is a formal power series in x with

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$[x^m]\{f(x)\} = a_m.$$

3.2 Equation for the generating functions

The recursive formulas of the previous chapter may now be considered as equations between the coefficients of the generating functions. Let us assume that $n+m \neq 0$. By Equation 2.3 we may write

$$\begin{aligned}\tilde{D}_{n,m} &= 2[x^n y^{m+1}]\{D(x,y)\} + [x^n y^{m-2}]\{D(x,y)^2\} \\ &= [x^n y^m]\{2yD(x,y) + y^2 D(x,y)^2\}.\end{aligned} \tag{3.3}$$

By taking the special case $n+m=0$ into account by setting $\tilde{D}_{0,0}=1$, we obtain the following equation

$$\begin{aligned}\tilde{D}(x,y) &= 1 + 2yD(x,y) + y^2 D(x,y)^2 \\ &= (1 + yD(x,y))^2\end{aligned} \tag{3.4}$$

for the formal power series $\tilde{D}(x,y)$.

Let us assume that $n \geq 1$. By Equation 2.6 we may write

$$\begin{aligned}\hat{D}_{n,m} &= [x^{n-1} y^{m-1}]\{D(x,y)\} - [x^{n-1} y^m]\{D(x,0)D(x,y)\} \\ &= [x^n y^m]\{xy^{-1}D(x,y) - xD(x,0)D(x,y)\}.\end{aligned} \tag{3.5}$$

The series on the RHS of the equation above does not involve zero order terms of x implying that the equation holds also for $n=0$. After we have subtracted the terms involving y^{-1} from the series on the RHS, we obtain

$$\begin{aligned}\hat{D}(x,y) &= xy^{-1}D(x,y) - xD(x,0)D(x,y) - xy^{-1}D(x,0) \\ &= x\left(y^{-1}\left(D(x,y) - D(x,0)\right) - D(x,0)D(x,y)\right).\end{aligned} \tag{3.6}$$

By Equations 3.2, 3.4 and 3.6 we have

$$D(x,y) = (1 + yD(x,y))^2 + x\left(y^{-1}\left(D(x,y) - D(x,0)\right) - D(x,0)D(x,y)\right),$$

which after multiplying both sides with y leads to the following equation for the generating function $D(x,y)$

$$y^3 D(x,y)^2 + \left(2y^2 + x - xD(x,0) - y\right) D(x,y) + y - xD(x,0) = 0. \tag{3.7}$$

Equation 3.7 is considered here as equations between the coefficients of two formal power series, where one of the series has all its coefficients equal to zero. The equations between the coefficients are the same as the equations given by the recursion formula 2.7. It follows that $D(x,y)$ is uniquely determined by Equation 3.7 and the initial condition $D_{0,0}=1$.

3.3 Solution to the equation

Before we shall start our search for a solution to the quadric equation 3.7, we shall first give a short introduction to formal power series and after that prove Lagrange's inversion theorem, a powerful tool in the field of analytic functions. In this connection, the theorem is extended to hold also for formal power series.

3.3.1 Formal power series

Definition 3.2. [Formal power series] Let us define the set $\mathbb{C}[[x]$ of formal power series in x over the complex numbers by

$$\mathbb{C}[[x]] = \left\{ \sum_{n=0}^{\infty} a_n x^n : a_n \in \mathbb{C} \text{ for every } n \right\}.$$

Let us equip $\mathbb{C}[[x]]$ with an addition

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

and a multiplication

$$\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n.$$

Then $(\mathbb{C}[[z]], +, \cdot)$ is a ring with the zero element equalling to the null series and the unit element equalling to the series where $a_0 = 1$ and $a_n = 0$ for $n \geq 1$.

Next we shall state couple of useful propositions that we are going to need in the proof of the Lagrange's inversion theorem. Although the proofs of the propositions are neither difficult nor long, we shall skip them in this connection. The proofs and more information about formal power series can be found for example from Wilf's book [15].

Proposition 3.3. *A formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$ has an unique multiplicative inverse $\frac{1}{f(x)} \in \mathbb{C}[[x]]$ if and only if $a_0 \neq 0$.*

Proposition 3.4. *A formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$ has an unique composition inverse $f^{-1}(x) \in \mathbb{C}[[x]]$ such that*

$$f^{-1}(f(x)) = x$$
$$f\left(f^{-1}(x)\right) = x$$

if and only if $a_0 = 0$ and $a_1 \neq 0$.

Theorem 3.5 (Lagrange's inversion theorem for formal power series)**.** *Assume that $f(y)$ and $g(y)$ are formal power series in y over the field $\mathbb{C}$, that is, $f(y), g(y) \in \mathbb{C}[[y]]$. Let us also assume that $g(0) = 1$. Then there exists an unique formal power series $y(x) \in x\mathbb{C}[[x]]$ that satisfies*

$$y(x) = xg(y(x)). \tag{3.8}$$

In addition, the formal power series $f(y(x))$ satisfies

$$[x^n]\{f(y(x))\} = \frac{1}{n}\left[y^{n-1}\right]\{f'(y)g(y)^n\} \tag{3.9}$$

when $n \geq 1$, and the constant term $[x^0]\{f(y(x))\}$ of the series is equal to $f(0)$.

Proof. Let us define a formal power series $x(y)$ by

$$x(y) = \frac{y}{g(y)}. \tag{3.10}$$

Since the constant term of $g(y)$ is one, it has a multiplicative inverse $1/g(y)$ in the ring $\mathbb{C}[[y]]$ of formal powers series (see Proposition 3.3). Hence $x(y)$ is a well-defined formal power series in y without a constant term. By the definition of multiplicative inverse, the constant term of $1/g(y)$ is also equal to one (see Definition 3.2). It follows that the coefficient of y in $x(y)$ is one and hence $x(y)$ has an unique composition inverse $x^{-1}(y) \in y\mathbb{C}[[y]]$ (see Proposition 3.4). Let us change the variable of the composition inverse from y to x and denote the resulting formal power series with $y(x)$. Now $y(x) \in x\mathbb{C}[[x]]$ and

$$\begin{aligned} x(y(x))g(y(x)) &= xg(y(x)) \\ &= y(x) \end{aligned}$$

completing the proof of the first claim of the theorem.

Since the second claim of the theorem is self-evident for $n = 0$, let us fix $n \geq 1$. Since $y(x)$ is without a constant term, we notice that the terms of $f(y)$ involving powers y^k for $k > n$ do not affect the LHS of Equation 3.9. The very same holds for the RHS of the equation. In addition, the terms of $g(y)$ involving powers y^k for $k > n-1$ do not affect the RHS of the equation. Hence, we may assume that $f(y)$ and $g(y)$ are polynomials by omitting the higher than n order terms of them.

By Equation 3.10 we have

$$\begin{aligned} \left[y^{n-1}\right]\{f'(y)g(y)^n\} &= \left[y^{n-1}\right]\{f'(y)\,(y/x(y))^n\} \\ &= \left[y^{-1}\right]\{f'(y)/x(y)^n\}. \end{aligned} \tag{3.11}$$

Since $g(0) = 1$ and g was assumed to be a polynomial, we know that $x(y)$ is analytic in some neighbourhood of zero. Let γ be a circle centred at zero and contained in that neighbourhood. By using the residue theorem to Equation 3.11 we obtain

$$\left[y^{-1}\right]\{f'(y)/x(y)^n\} = \frac{1}{2\pi i}\oint_{\gamma}\frac{f'(y)}{x(y)^n}\,dy. \tag{3.12}$$

Since

$$x'(0) = 1,$$

the function $x(y)$ is locally bijective at zero and hence it has an analytic inverse function, which itself is defined in some neighbourhood of zero. Let us assume that γ was chosen in such a way that its image $\tilde{\gamma}$ under x is contained in that neighbourhood. Now we may change the integration variable in Equation 3.12 from y to x, and by using the residue theorem to the analytic derivative of $f(y(x))$ conclude

$$\begin{aligned} \left[y^{n-1}\right]\{f'(y)g(y)^n\} &= \frac{1}{2\pi i}\oint_{\tilde{\gamma}}\frac{f'(y(x))y'(x)}{x^n}\,dx \\ &= \frac{1}{2\pi i}\oint_{\tilde{\gamma}}\frac{1}{x^n}\frac{d}{dx}f(y(x))\,dx \\ &= \left[x^{n-1}\right]\left\{\frac{d}{dx}f(y(x))\right\} \\ &= n[x^n]\{f(y(x))\}. \end{aligned}$$

□

3.3.2 Solution

Theorem 3.6 (Number of rooted triangulations)**.** *The number $D_{n,m}$ of triangulations of type $[n, m]$ of the closed disk satisfies*

$$D_{n,m} = \frac{2(2m+3)!(4n+2m+1)!}{(m+2)!m!n!(3n+2m+3)!}. \tag{3.13}$$

Proof. Let us first consider the quadratic equation

$$y^3L(x,y)^2 + \left(2y^2 + x - xL(x,0) - y\right) L(x,y) + y - xL(x,0) = 0 \tag{3.14}$$

and assume that we have found a solution $L(x,y)$, which is analytic at the origin. In that case, the coefficients of the power series expansion of $L(x,y)$ about the origin satisfy Equation 2.7 and by uniqueness of the solution the expansion has to be the generating function $D(x,y)$.
Let us set

$$\begin{aligned} x &= rs^3 \\ s &= 1 - r \\ L(x,0) &= s^{-3}(1-2r) \end{aligned} \tag{3.15}$$

and substitute in Equation 3.14. After that, the discriminant of Equation 3.14 writes

$$\begin{aligned} &\left(2y^2 - y + rs^3 - yr(1-2r)\right)^2 - 4y^3\left(y - r(1-2r)\right) \\ =& -4y^3 + (1 + 2r - 3r^2 - 4r^3 + 4r^4 + 4rs^3)y^2 \\ &+ (-2rs^3 - 2r^2s^3 + 4r^3s^3)y + r^2s^6. \end{aligned}$$

By using the relationship between r and s appropriately, the latter form above can be written as

$$\begin{aligned} &-4y^3 + (s^2 + 8rs^2)y^2 + (-2rs^4 - 4r^2s^4)y + r^2s^6 \\ =& (y^2 - 2rs^2y + r^2s^4)(s^2 - 4y) \\ =& (y - rs^2)^2(s^2 - 4y). \end{aligned}$$

Hence

$$2y^3L(x,y) = -2y^2 - rs^3 + ys(1+2r) \pm (ys - rs^3)\left(1 - \frac{4y}{s^2}\right)^{\frac{1}{2}}, \tag{3.16}$$

where by expanding the square root term about the origin

$$
\begin{aligned}
(ys - rs^3)\left(1 - \frac{4y}{s^2}\right)^{\frac{1}{2}} &= (ys - rs^3)\left(1 - 2\sum_{n=0}^{\infty}\frac{(2n)!}{n!(n+1)!}\left(\frac{y}{s^2}\right)^{n+1}\right) \\
&= ys - rs^3 - 2y^2s^{-1} + 2r(ys + y^2s^{-1}) \\
&\quad - 2\sum_{n=1}^{\infty}\frac{(2n)!}{n!(n+1)!}y^{n+2}s^{-2n-1} \\
&\quad + 2r\sum_{n=2}^{\infty}\frac{(2n)!}{n!(n+1)!}y^{n+1}s^{-2n+1} \\
&= -2y^2 - rs^3 + ys(1+2r) \\
&\quad - 2\sum_{m=0}^{\infty}\frac{(2m+2)!}{(m+1)!(m+2)!}y^{m+3}s^{-2m-3} \\
&\quad + 2\sum_{m=0}^{\infty}\frac{r(2m+4)!}{(m+2)!(m+3)!}y^{m+3}s^{-2m-3}.
\end{aligned}
$$

If the plus sign were to be selected in Equation 3.16, then the power series of $L(x, y)$ would involve negative powers of y and thus it could not be the solution we are looking for that is analytic at the origin. By selecting the minus sign in Equation 3.16 we obtain

$$
\begin{aligned}
L(x, y) &= \sum_{m=0}^{\infty}\frac{(2m+2)!(m+3)(m+2) - r(2m+4)!}{(m+3)!(m+2)!}s^{-2m-3}y^m \\
&= 2\sum_{m=0}^{\infty}\frac{(2m+1)!}{(m+3)!m!}\left((m+3) - 2r(2m+3)\right)s^{-2m-3}y^m \qquad (3.17) \\
&=: \sum_{m=0}^{\infty}a_m y^m.
\end{aligned}
$$

By Equation 3.15 we may choose the inverse $s(x)$ of $x(s)$ in such a way that $s(0) = 1$. Now when applying the ratio test to the coefficients a_m we obtain

$$
\left|\frac{a_{m+1}}{a_m}\right| = \frac{(2m+3)(2m+2)}{(m+4)(m+1)}\left|\frac{m+4-2(2m+5)r}{m+3-2(2m+3)r}\right|\left|s^{-2}\right|.
$$

If x is chosen small enough, we may assume that $r = 1 - s < \frac{1}{4}$. Then the fractional term on the RHS inside the absolute value operator is positive and can be written as

$$\frac{1+\frac{4}{m}-2\left(2+\frac{5}{m}\right)r}{1+\frac{3}{m}-2\left(2+\frac{3}{m}\right)r}.$$

Hence

$$\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|=\frac{4}{s^2}$$

and thus $L(x,y)$ is analytic in some neighbourhood of the origin.

Now we do know that $L(x,y)=D(x,y)$ and thus we may write Equation 3.17 as

$$D(x,y)=2\sum_{m=0}^{\infty}\frac{(2m+1)!}{(m+3)!m!}\left((m+3)s^{-2m-3}-2(2m+3)xs^{-2m-6}\right)y^m, \tag{3.18}$$

where the coefficients of y^m are equal to $D_{.m}(x)$ of Equation 3.1. We shall use Lagrange's inversion theorem (3.5) with the following choice of formal power series

$$\begin{aligned} g(r) &= (1-r)^{-3} \\ f(r) &= (1-r)^{-t}, \quad t\geq 1 \\ r &= x(1-r)^{-3} \end{aligned}$$

to be able to express the coefficients in terms of x. From Lagrange's theorem it is visible that $r(x)=0$ when $x=0$, which is consistent with the earlier made choice of $s(x)$. Theorem 3.5 yields

$$\begin{aligned} [x^n]\{(1-r(x))^{-t}\} &= \frac{1}{n}[r^{n-1}]\{t(1-r)^{-t-1}(1-r)^{-3n}\} \\ &= \frac{t}{n(n-1)!}\left[\frac{d^{n-1}}{dr^{n-1}}(1-r)^{-t-1-3n}\right]\Bigg|_{r=0} \\ &= \frac{t}{n!}\frac{(4n+t-1)!}{(3n+t)!} \quad \text{for } n\geq 1 \end{aligned} \tag{3.19}$$

and that the constant term of $(1-r)^{-t}$ is equal to $f(0)=1$. Hence

$$\begin{aligned} s(x)^{-t} &= 1+\sum_{n=1}^{\infty}\frac{t}{n!}\frac{(4n+t-1)!}{(3n+t)!}x^n \\ &= \sum_{n=0}^{\infty}\frac{t}{n!}\frac{(4n+t-1)!}{(3n+t)!}x^n. \end{aligned}$$

Now by Equation 3.18

$$\begin{aligned} D_{.m}(x) &= \frac{2(2m+1)!}{(m+3)!m!}\Bigg((2m+3)(m+3)\sum_{n=0}^{\infty}\frac{(4n+2m+2)!}{n!(3n+2m+3)!}x^n \\ &\quad -2(2m+6)(2m+3)\sum_{k=0}^{\infty}\frac{(4k+2m+5)!}{k!(3k+2m+6)!}x^{k+1}\Bigg) \\ &= \frac{2(2m+1)!(2m+3)}{(m+2)!m!}\Bigg(\sum_{n=0}^{\infty}\frac{(4n+2m+2)!}{n!(3n+2m+3)!}x^n \\ &\quad -4\sum_{n=1}^{\infty}\frac{(4n+2m+1)!}{(n-1)!(3n+2m+3)!}x^n\Bigg), \end{aligned} \tag{3.20}$$

where the equality follows from the change of the index of summation of the second sum from k to $n = k+1$. The two sums of the latter expression above can be combined into the following form

$$\begin{aligned} &\frac{1}{(2m+3)} + \sum_{n=1}^{\infty}\frac{(4n+2m+1)!(2m+2)}{n!(3n+2m+3)!}x^n \\ &= \sum_{n=0}^{\infty}\frac{(4n+2m+1)!(2m+2)}{n!(3n+2m+3)!}x^n. \end{aligned}$$

Thus Equation 3.20 gives

$$D_{.m}(x) = \frac{2(2m+3)!}{(m+2)!m!}\sum_{n=0}^{\infty}\frac{(4n+2m+1)!(2m+2)}{n!(3n+2m+3)!}x^n,$$

where the coefficients of x^n are equal to $D_{n,m}$ yielding

$$D_{n,m} = \frac{2(2m+3)!(4n+2m+1)!}{(m+2)!m!n!(3n+2m+3)!}.$$

□

Chapter 4

Circle packings

In this chapter we are first going to introduce circle packings and terminology related to them. We shall start from the situation where a circle packing can lie in any orientable surface with a metric. After a while, we shall restrict our considerations to the complex plane $\mathbb{C}$, Poincaré disk $\mathbb{D}$ and Riemann sphere $\mathbb{P}$ as representatives of Euclidean, hyperbolic and spherical geometries respectively. Although some illustrative examples in the complex plane will be provided, our main focus lies in the Poincaré disk model as a representative of 2-dimensional hyperbolic geometry. The main result of the chapter is the existence of the maximal circle packing in the Poincaré disk (Theorem 4.25). The structure and the proofs of the chapter follow closely Stephenson's book [13], but also the papers [6], [11] and [12] were of assistance. All the figures containing configurations of circles are made by using Kenneth Stephenson's CirclePack program [10].

4.1 Circle packings in general

We shall define circle packings through abstract 2-complexes presenting the combinatorics of circle packings. Although it would be possible to define circle packings for a wider range of abstract 2-complexes, we shall restrict our considerations to 2-complexes that satisfy the requirements of Proposition 2.6. That is, every complex $\mathscr{K}$ is assumed to represent a triangulation of a topological oriented surface. The restriction guarantees the corresponding circle packings more of such a structure that is easily dictated by one's intuition.

Definition 4.1 (Circle packings)**.** Let $\mathcal{G}$ be an oriented surface with a metric. Then a collection $P = \{c_\alpha\} \subset \mathcal{G}$ of circles is said to be a circle packing for a 2-complex $\mathscr{K}$ if

(1) For every vertex $v \in \mathscr{K}$ there exists an unique circle $c_v \in P$.

(2) If $\langle vu \rangle$ is an edge of $\mathscr{K}$, then the circles c_v and c_u are externally tangent.

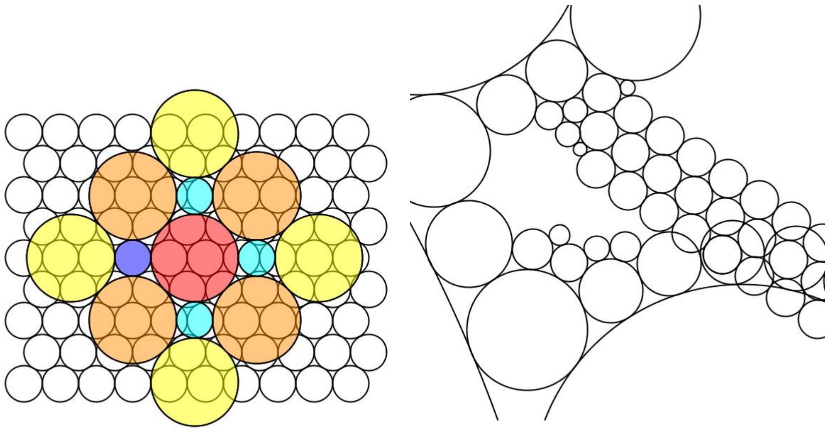
(3) If $\langle vuw \rangle$ is a face of $\mathscr{K}$, then the circles c_v, c_u and c_w are mutually externally tangent to each other. That is, the circles form a triple in $\mathcal{G}$. In addition, if $\langle vuw \rangle^*$ is positively oriented in $\mathscr{K}$ then the corresponding triple $\langle c_v c_u c_w \rangle$ is positively oriented in $\mathcal{G}$.

Notice that in the definition, the circles of the packing P are not required to have mutually disjoint interiors. If v and u are vertices of $\mathscr{K}$ and there is no edge $\langle vu \rangle$ in $\mathscr{K}$, in general, nothing guarantees that the corresponding circles c_v and c_u have mutually disjoint interiors. However, if there is no overlapping circles in P, we say that P is an univalent circle packing. We shall get back to this subject after we have introduced some terminology of circle packings.

Definition 4.2. Let us assume that v and u are two vertices of a 2-complex $\mathscr{K}$. Let us also assume that P is a circle packing for $\mathscr{K}$ and that c_v and c_u are the circles of P for the vertices v and u. If $\langle uv \rangle$ is an edge of $\mathscr{K}$, we say that v and u (c_v and c_u) are neighbours and write $v \backsim u$ ($c_v \backsim c_u$). The neighbours of the vertex v can be presented in such an ordered list $(v_1, ..., v_n)$ that $v_j \backsim v_{j+1}$ for $j \in \{1, ..., n-1\}$. We shall call $F_v = \{v; v_1, ..., v_n\}$ the combinatorial flower of the vertex v and the corresponding geometrical configuration $F_{c_v} = \{c_v; c_{v_1}, ..., c_{v_n}\}$ of circles the geometrical flower of the circle c_v. The number n is called the degree of the vertex v as well as the degree of the circle c_v. If v is an interior vertex, then $v_n \backsim v_1$ and we say that the flowers F_v and F_{c_v} are closed.

If a circle packing P has an interior circle c_v whose flower wraps more than once around c_v, we say that P is a branched circle packing and c_v is a branch circle. If the flower of a branch circle c_v wraps $n+1$ times around it, we say that n is the branch order of c_v. An example of a branched circle packing in the complex plane is provided in Subfigure 4.1a. The packing of the figure has a branch circle of branch order one coloured with lilac. The degree of the branch circle is 10. The small circles coloured with light blue have a degree of four and the big circle coloured with red has a degree of eight. If the pattern of the packing would be extended through the whole plane, all the big circles would have a degree of eight and all the small circles (except the branch circle) would have either a degree of six or four.
A circle packing having no branch circles is said to be locally univalent. An example of a locally univalent circle packing is provided in Subfigure 4.1b. In the packing of the figure, two different parts of the packing are overlapping,

although the circles of single flowers are non-overlapping. However, since branching was defined only for the interior circles, in general, there might be overlapping of circles in the flower of a boundary circle, even if the circle packing is locally univalent.



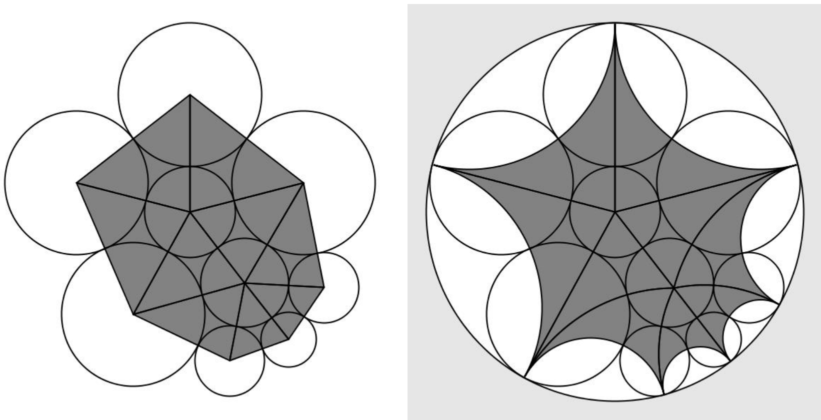
(a) A branched circle packing. (b) A locally univalent circle packing.

Figure 4.1: Examples of non-univalent circle packings.

Definition 4.3 (Carriers)**.** Let us assume that P is a circle packing for a 2-complex $\mathscr{K}$ in $\mathcal{G}$. We may construct a geometrical structure called the carrier of P by connecting centres of the neighbouring circles of P with geodesic lines of $\mathcal{G}$. We shall denote the carrier of P with carr(P). If P is univalent, then carr(P) provides us with an embedding of $\mathscr{K}$ in $\mathcal{G}$ and we shall treat carr(P) as a geometrical realization of $\mathscr{K}$.

If a circle packing P is univalent, then the circle centres correspond to the 0-simplices, the geodesics between the centres of neighbouring circles correspond to the 1-simplices and the triangles formed by connecting the centres of three mutually neighbouring circles with geodesics correspond to the 2-simplices of $\mathscr{K}$.
Examples of carriers in $\mathbb{C}$ and $\mathbb{D}$ are provided in Figure 4.2. Both circle packings of the figure are univalent circle packings for a 2-complex $\mathscr{K}$ having nine vertices, from which seven are boundary vertices.

From this point on we shall restrict our considerations to the three constant curvature geometries, namely Euclidean, hyperbolic and spherical. We shall use the complex plane $\mathbb{C}$, the Poincaré disk $\mathbb{D}$ and the Riemann sphere $\mathbb{P}$ as representatives of the three geometries respectively, the hyperbolic disk



(a) Carrier in $\mathbb{C}$ (b) Carrier in $\mathbb{D}$

Figure 4.2: Carriers

$\mathbb{D}$ being our main subject of interest. The Riemann sphere consists of the points of the three-dimensional unit sphere and it can be identified with the extended complex plane $\bar{\mathbb{C}}$ through the stereographic projection. Some basic properties of the spherical geometry and the Riemann sphere can be found for example from Stephenson's book [13].
The Poincaré disk model uses the points of the Euclidean open unit disk as the points of the geometry. We shall provide more properties of the hyperbolic disk in the beginning of the next section.

Every circle packing comes with the radii representing the geometrical properties of the packing. The question is, if we have a set R of potential radii in a specific geometry and a 2-complex $\mathscr{K}$, when do they fit together in such a way that there exits a circle packing obeying the combinatorics of $\mathscr{K}$ and the geometry of R? To study this question, we shall first make some observations about radii in our model geometrical spaces and after that provide couple of new definitions. In the case of a simply connected complex $\mathscr{K}$, the question will be answered in Theorem 4.7.
Let us consider a circle packing P in one of our model geometrical spaces. If the circle packing P lies in $\mathbb{C}$, then the radii of the circles of P are strictly positive real numbers. If P lies in $\mathbb{D}$, then the radii are strictly positive real numbers, but also infinite radii is allowed for the boundary circles of P. If P lies in $\mathbb{P}$, then the radii lie in the open interval $(0, \pi)$.

Definition 4.4 (Labels)**.** Let $\mathbb{G}$ be either $\mathbb{C}, \mathbb{D}$ or $\mathbb{P}$ and let $\{v_1, v_2, ...\}$ be the vertex set of a 2-complex $\mathscr{K}$. Assume that $R = \{r_1, r_2, ...\}$ is a set of real numbers that qualify as radii in $\mathbb{G}$ and that R associates every vertex

$v_j \in \mathscr{K}$ with the real number $r_j = R(v_j)$. Then we say that R is a label for $\mathscr{K}$ in $\mathbb{G}$ and that $\mathscr{K}(R)$ is a labeled complex in $\mathbb{G}$.
Let us assume that a label R consists of the radii of a circle packing P for $\mathscr{K}$, meaning that $R(v) = \text{radius}(c_v)$ for every vertex $v \in \mathscr{K}$. In this case we write $P \longleftrightarrow \mathscr{K}(R)$.

Let us assume that r_1, r_2 and r_3 are labels in $\mathbb{G}$ for vertices v_1, v_2 and v_3 respectively. Then there exits a triple (in $\mathbb{P}$ also the condition $r_1+r_2+r_3 \leq 2\pi$ has to be satisfied) $\langle c_{v_1} c_{v_2} c_{v_3} \rangle$ of circles in $\mathbb{G}$ having the radii determined by the labels. The triple is unique up to the isometries of $\mathbb{G}$. The geodesics between the centres of the circles of the triple form a hyperbolic triangle, which angles can be calculated with the law of cosines of $\mathbb{G}$. Let us denote the angle at the centre of the circle c_{v_1} with α, where α is given by the angle map $\alpha(r_1, r_2, r_3)$ of the geometry of the labels. The explicit formula for the angle map in the hyperbolic geometry is provided by Lemma 4.8. For the other two geometries the formula for the angle sum map can be found from Stephenson's book [13].
A face of a labelled complex $\mathscr{K}(R)$ can be identified with the triangle formed by a triple, whether there exists a circle packing $P \longleftrightarrow \mathscr{K}(R)$ or not. Therefore we may discuss the angles and the area of a face.

Definition 4.5 (The angle sum map)**.** Let $\mathscr{K}(R)$ be a labelled complex and let v be a vertex of $\mathscr{K}$. Let us denote the vertex set of $\mathscr{K}$ with $\mathscr{K}^0$. Consider all the faces $\langle vuw \rangle$ of the flower F_v of v. The angle sum map $\theta_R : \mathscr{K}^0 \to [0, \infty)$

$$\theta_R(v) = \sum_{\langle vuw \rangle} \alpha(R(v), R(u), R(w))$$

associates v with the sum of the angles at v over these faces.

It is easy to see that there exists a geometrical flower F_{c_v} for the combinatorial flower F_v of an interior vertex v in the metric space of R if and only if $\theta_R(v) = 2\pi$. In this case, we say that the packing condition is satisfied at v.

Definition 4.6 (Packing labels)**.** Let $\mathscr{K}(R)$ be a labelled complex. If for every interior vertex $v \in \mathscr{K}$ there exists $\beta_v \in \mathbb{N} \cup \{0\}$ such that $\theta_R(v) = 2\pi(1 + \beta_v)$, we say that R is a packing label for $\mathscr{K}$. If $\beta_v \geq 1$, we say that R has a branch point at v and that β_v is the branch order of v. If R has no branch points, then R is an unbranched packing label for $\mathscr{K}$. Otherwise R is a branched packing label for $\mathscr{K}$.

If R is an unbranched packing label for $\mathscr{K}$ and there exists a circle packing P such that $P \longleftrightarrow \mathscr{K}(R)$, then P is locally univalent. That is, for every interior flower the related circles are non-overlapping, but there still might be overlapping of circles between different flowers of the packing and in the flowers of the boundary circles.
It would be tempting to assume that if R is a packing label for $\mathscr{K}$, then there would exist a circle packing $P \longleftrightarrow \mathscr{K}(R)$ in the metric space of R. R being a packing label is definitely a necessary condition for the existence of such a circle packing, but it is not sufficient in all the cases. However, if $\mathscr{K}$ is assumed to be simply connected, we shall show that being a packing label is also a sufficient condition for the existence of a circle packing $P \longleftrightarrow \mathscr{K}(R)$.

Theorem 4.7. *Let $\mathscr{K}(R)$ be a labeled complex where $\mathscr{K}$ is simply connected and let $\mathbb{G}$ be either $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{P}$ corresponding to the geometry of R. Then there exists a circle packing P for $\mathscr{K}$ in $\mathbb{G}$ with $P \longleftrightarrow \mathscr{K}(R)$ if and only if R is a packing label for $\mathscr{K}$. In addition, P is unique up to the isometries of $\mathbb{G}$.*

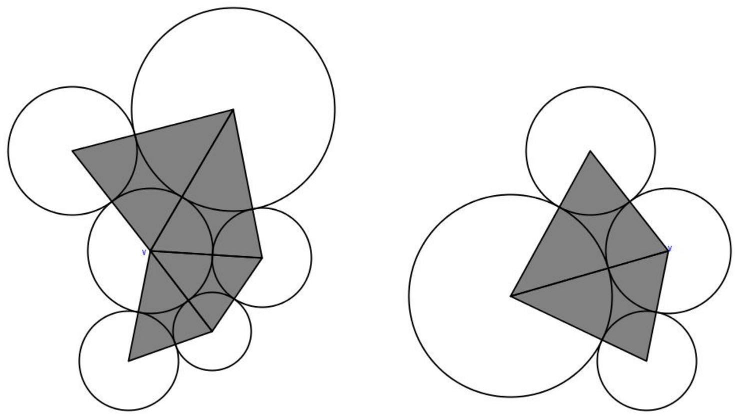
Proof. The forward implication is self-evident. Let us therefore assume that R is a packing label for $\mathscr{K}$ in $\mathbb{G}$. Let us also assume that we have an unlocated circle c_v with radius$(c_v) = R(v)$ for every vertex $v \in \mathscr{K}$. Our aim is to place these circles in such a way that they form a circle packing for $\mathscr{K}$ in $\mathbb{G}$.

First, choose two arbitrary vertices $v_1, v_2 \in \mathscr{K}$ such that $v_1 \backsim v_2$ and place the corresponding circles c_1 and c_2 at some arbitrary location in $\mathbb{G}$ in such a way that they will be externally tangent to each other. Then, select a vertex $v_3 \in \mathscr{K}$ such that $v_3 \backsim v_1$ and $v_3 \backsim v_2$. Place the circle c_3 for the vertex v_3 externally tangent to the circles c_1 and c_2 in such a way that the triple $\langle c_1c_2c_3 \rangle$ follows the orientation of $\langle v_1v_2v_3 \rangle^*$. Since the orientation of the triple $\langle c_1c_2c_3 \rangle$ is fixed, the location for c_3 is unique and can be calculated with the law of cosines of the particular geometry $\mathbb{G}$.
The face $f_0 = \langle v_1v_2v_3 \rangle$ determines a triangle t_0 in $\mathbb{G}$ through the geodesics between the centres of the corresponding triple. Now, if two circles of a face $f_k \in \mathscr{K}$ are already placed, then the placing of the third circle (by using the law of cosines) corresponds to the placing of the triangle t_k associated with the face f_k. Therefore we may concentrate on placing triangles instead of circles.
Consider now a face $f_n \in \mathscr{K}$, $f_n \neq f_0$ and a chain (Definition 2.3) $\Gamma = \{f_0, f_1, ..., f_n\}$ from f_0 to f_n. Connectedness of $\mathscr{K}$ guarantees that there exists such a chain. Since f_1 shares an edge with f_0, the already placed triangle t_0 determines an unique location for the triangle t_1 associated with the face

f_1. By continuing placing triangles, the location for t_n will be uniquely determined by the triangle t_0 and the chain Γ. We say that the location of t_n is obtained from the location of the base triangle t_0 by a development along Γ.

Next we want to show that the location of t_n is independent from the choice of the chain Γ. Let Γ' be an another chain from f_0 to f_n and let $-\Gamma' \circ \Gamma$ be the concatenated chain leading from f_0 to f_n along Γ and back from f_n to f_0 along $-\Gamma'$. The developments along Γ and Γ' will place t_n at the same location if and only if the development along $-\Gamma' \circ \Gamma$ will place t_0 back to its original location. Therefore we may concentrate to study closed chains of form $\Gamma = \{f_0, f_1, ..., f_0\}$.
Consider a subchain $\gamma = \{f_j, ..., f_k\}$ of such a closed chain Γ and assume that γ is local (Definition 2.3) at a vertex v. Let us form a local modification Γ' of Γ by replacing γ in Γ with some other local subchain $\gamma' = \{f_j, ..., f_k\}$ at v. The fact that R is a packing label guarantees that the triangles t_j and t_k associated with the faces f_j and f_k will be placed identically whether we are developing along Γ or Γ'. Hence local modifications do not affect the placement of the last triangle of a development. Since every closed chain of a simply connected complex $\mathscr{K}$ is homotopic to the null chain $\Gamma_0 = \{f_0\}$ through a finite number of local modifications, we may conclude that developments along closed chains will place the last triangle identically with the first one. Hence the placement of triangles is independent of the choice of the chain, along which we are developing. This means that after placing a triangle for every face of $\mathscr{K}$, the resulting configuration of circles will depend only on the placement of the original pair (c_1, c_2) of circles. Note also that the choice for vertices v_1 and v_2 does not affect the result, only the placement of the corresponding circles. Hence the result of our construction is unique up to the rigid motion, that is, up to the isometries of $\mathbb{G}$.
An example of a local modification is illustrated in Figure 4.3. In Subfigure 4.3a is presented an interior circle c_v with five of its six petals and a local chain at v. In Subfigure 4.3b is presented the circle c_v with three of its petals and an another local chain at v. The two chains have the same first and last face and thus they are local modifications of each other.

We still need to convince ourselves that what we have constructed is a circle packing for $\mathscr{K}$. The first requirement of Definition 4.1 will be clearly satisfied. Consider now an arbitrary vertex v of $\mathscr{K}$ and a chain from the base face f_0 to some of the faces in which v belongs to. We may extend the chain through the whole flower of v and a development along this extended chain will give the circle c_v for v all the tangencies described in $\mathscr{K}$. The orientation condition of Definition 4.1 gets satisfied by choosing correctly the location of



(a) (b)

Figure 4.3: A local modification at v.

the third placed circle. We may now conclude that we have formed a circle packing P in $\mathbb{G}$ with $P \longleftrightarrow \mathscr{K}(R)$. $\square$

4.2 Existence of the maximal circle packing in the hyperbolic disk

We shall begin this section by introducing some basic properties of the Poincaré disk. After that, we shall derive some important results for labelled flowers that are used in the proof of the main theorem (4.15) of the chapter. More information about the Poincaré disk can be found for example from Anderson's book [1].

4.2.1 Monotonicities

Lemma 4.8 (Properties of the Poincaré disk $\mathbb{D}$)**.**

Set of points:	$\mathbb{D} = \{z \in \mathbb{C} : \lvert z\rvert < 1\}$
Arc-length element:	$ds = \dfrac{2}{1-\lvert z\rvert^2}\lvert dz\rvert$
Triangle area:	$\mathrm{area}(t) = \pi - \alpha - \beta - \gamma$
Law of cosines:	$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$
$Aut(\mathbb{D})$ = $Isom(\mathbb{D})$=	$\left\{\Phi(z) = e^{i\theta}\dfrac{z+z_0}{1+\bar{z}_0 z} : \theta \in \mathbb{R}, z_0 \in \mathbb{D}\right\}$,

where $\lvert dz\rvert$ is the arc-length element of the Euclidean plane and α, β and γ denote the angles of a hyperbolic triangle t. In the law of cosines a, b and c are the lengths of the sides of a hyperbolic triangle and γ denotes the angle between sides of lengths a and b. In addition, $Aut(\mathbb{D})$ and $Isom(\mathbb{D})$ denote the group of automorphisms and the group of isomorphisms of the Poincaré disk respectively.

A circle in $\mathbb{D}$ is also a circle in the Euclidean plane and a circle of the Euclidean plane inside the unit disk is also a circle in $\mathbb{D}$. The geodesics of $\mathbb{D}$ consists of diameters of the disk and arcs of the Euclidean circles that are orthogonal to the unit circle. Differing from the Euclidean geometry, also infinite is a possible radius of a circle in $\mathbb{D}$.

Definition 4.9 (Horocycles)**.** Let us assume that c is a circle in $\mathbb{D}$ having an infinite radius. Then c is internally tangent to the boundary $\partial\mathbb{D}$ of the unit disk and its centre is defined to be located at the point of tangency. We shall call the circle c a hororycle.

For example, in the circle packing of Figure 4.2b all the seven boundary circles are horocycles.

Lemma 4.10 (The angle map in $\mathbb{D}$)**.** *Let t be a hyperbolic triangle in $\mathbb{D}$ formed by connecting the centres of a triple with geodesic lines. Let us denote the radii of the circles with r_1, r_2 and r_3. The angle α of t located at the centre of the circle with the radius r_1 is then given by*

$$\alpha(r_1, r_2, r_3) = \begin{cases} \arccos\left(\frac{\cosh(r_1+r_2)\cosh(r_1+r_3)-\cosh(r_2+r_3)}{\sinh(r_1+r_2)\sinh(r_1+r_3)}\right), & r_1, r_2, r_3 \in (0,\infty) \\ \arccos\left(\frac{\cosh(r_1+r_2)-e^{r_2-r_1}}{\sinh(r_1+r_2)}\right), & r_1, r_2 \in (0,\infty), r_3 = \infty \\ \arccos(1-2e^{-2r_1}), & r_1 \in (0,\infty), r_2 = r_3 = \infty \\ 0, & r_1 = \infty. \end{cases} \tag{4.1}$$

The first case of the formula follows directly from the hyperbolic law of cosines of Lemma 4.8. The second case is the limit of the first case as $r_3 \to \infty$, the third case is the limit of the second case as $r_2 \to \infty$ and the final case equals to the limits of the three other cases as $r_1 \to \infty$.

Lemma 4.11 (Monotonicity in triangles)**.** *Let t be a hyperbolic triangle in $\mathbb{D}$ formed by connecting the centres of a triple with geodesic lines. Let us denote the radii of the circles in the triple with r_1, r_2 and r_3, and the angles at the centres of the circles respectively with α_1, α_2 and α_3. If r_1 is assumed to be finite, then α_1 is continuous and strictly decreasing in r_1. In addition, α_1 is continuous and strictly increasing in those of the radii r_2 and r_3 that are finite.*
Furthermore, $\text{area}(t)$ *is continuous and strictly increasing in the radii that are finite.*

Proof. **Case 1:** $r_1, r_2, r_3 \in (0, \infty)$

By using the definitions of the hyperbolic trigonometric functions to Equation 4.1 we obtain

$$\begin{aligned}
&\alpha_1(r_1, r_2, r_3) \\
&= \arccos\left(\frac{(e^{r_1+r_2} + e^{-r_1-r_2})(e^{r_1+r_3} + e^{-r_1-r_3}) - 2(e^{r_2+r_3} + e^{-r_2-r_3})}{(e^{r_1+r_2} - e^{-r_1-r_2})(e^{r_1+r_3} - e^{-r_1-r_3})}\right) \\
&= \arccos\left(\frac{(1 + e^{2(r_1+r_2)})(1 + e^{2(r_1+r_3)}) - 2e^{2r_1}(1 + e^{2(r_2+r_3)})}{(e^{2(r_1+r_2)} - 1)(e^{2(r_1+r_3)} - 1)}\right).
\end{aligned}$$

Let us define a function $g_1(x, y, z)$ by making substitutions $x = e^{2r_1}, y = e^{2r_2}$ and $z = e^{2r_3}$ in the latter form above obtaining

$$\begin{aligned}
g_1(x, y, z) &= \arccos\left(\frac{(1 + xy)(1 + xz) - 2x(1 + yz)}{(xy - 1)(xz - 1)}\right) \qquad (4.2) \\
\frac{\partial g_1}{\partial x}(x, y, z) &= -\frac{(x^2yz - 1)}{(xy - 1)(xz - 1)}\sqrt{\frac{(y - 1)(z - 1)}{x(x - 1)(xyz - 1)}} \\
\frac{\partial g_1}{\partial y}(x, y, z) &= \frac{1}{xy - 1}\sqrt{\frac{x(x - 1)(z - 1)}{(y - 1)(xyz - 1)}}.
\end{aligned}$$

Since $x, y, z > 1$, the function g_1 is continuous with respect to all its variables. By the same argument, the derivative with respect to x is strictly negative and the derivative with respect to y is strictly positive. By symmetry, the derivative with respect to z is strictly positive. By using the chain rule and the definitions of x, y and z, the monotonicity of the angle α_1 in the radii follows.

Since area$(t) = \pi - \alpha_1 - \alpha_2 - \alpha_3$, continuity of the area of the triangle t is clear. However, since an increase in one of the radii increases one of the angles and decreases the other two, it is not obvious what is the total effect of changing a radius on the area of t. Of course, this could be found out by differentiating the formula of the area, but we shall provide here a more intuitive way for proving the monotonicity claim of the area.
Let us consider a triple $\langle c_1 c_2 c_3 \rangle$ of circles with a label $\{r_1 r_2 r_3\}$ in $\mathbb{D}$. An increase in the radius r_1 moves the centre of c_1 further away from the other centres. This increases the angle α_2 at the centre of c_2 and the angle α_3 at the centre of c_3, as proved above. The original triangle will be strictly contained in the new one providing us with monotonicity of the area of a hyperbolic triangle.
An illustrative example is provided in Figure 4.4. The original triple and the original triangle is coloured with black. The radius of the circle c_1 is increased. The circle c_1, after the increase in its radius, and the new triangle are coloured with red.

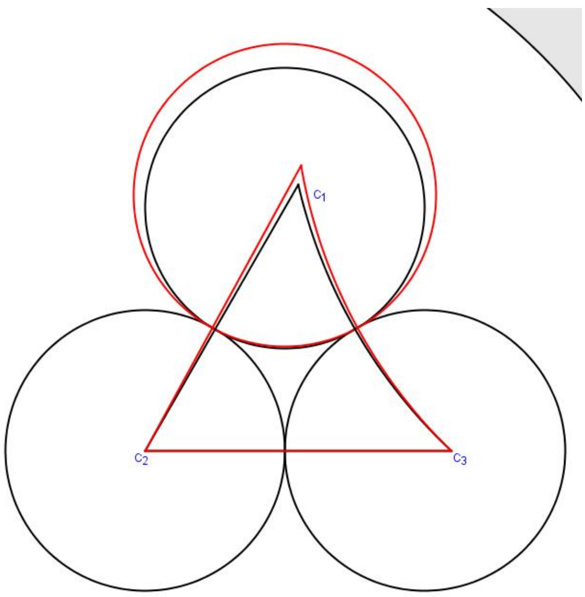


Figure 4.4: Monotonicity of the area of a triangle.

Case 2: $r_1, r_2 \in (0, \infty), z = \infty$

Let us define a function $g_2(x, y)$ by

$$\begin{aligned} g_2(x,y) &= \lim_{z\to\infty} g_1(x,y,z) \\ &= \lim_{z\to\infty} \arccos\left(\frac{(1+xy)(\frac{1}{z}+x) - 2x(\frac{1}{z}+y)}{(xy-1)(x-\frac{1}{z})}\right) \\ &= \arccos\left(\frac{xy-2y+1}{xy-1}\right). \end{aligned} \tag{4.3}$$

The first derivatives of g_2 are

$$\begin{aligned} \frac{\partial g_2}{\partial x}(x,y) &= -\sqrt{\frac{y(y-1)}{(x-1)(xy-1)^2}} \\ \frac{\partial g_2}{\partial y}(x,y) &= \sqrt{\frac{x-1}{y(y-1)(xy-1)^2}}. \end{aligned}$$

By the same arguments as in the first case, the claim of the lemma holds.

Case 3: $r_1 \in (0, \infty), y = z = \infty$

Let us define a function $g_3(x)$ by

$$\begin{aligned} g_3(x) &= \lim_{y\to\infty} g_2(x,y) \\ &= \lim_{y\to\infty} \arccos\left(\frac{x-2+\frac{1}{y}}{x-\frac{1}{y}}\right) \\ &= \arccos(1-2x^{-1}). \end{aligned} \tag{4.4}$$

The first derivative of g_3 is

$$\frac{\partial g_3}{\partial x}(x) = -\frac{1}{\sqrt{x^2(x-1)}},$$

proving the monotonicity claim of α_1. Since $\alpha_2, \alpha_3 = 0$, monotonicity of the area can be seen directly from the equation $\text{area}(t) = \pi - \alpha_1$. □

Let us assume that F_v is the flower of a vertex v and that R is a label for F_v. Then we shall denote the sum of the areas of the faces of the flower F_v with $\text{area}(F_v)$.

Corollary 4.12 (Monotonicity in flowers). *Let $F_v = \{v; v_1, ..., v_n\}$ be the combinatorial flower of a vertex v and let $R = \{r; r_1, ..., r_n\}$ be a hyperbolic label for F_v with $r < \infty$. Then the angle sum $\theta_R(v)$ at v is continuous and strictly decreasing in r, and continuous and strictly increasing in the radii r_i, $i = 1, 2, ..., n$ that are finite. In addition,* area(F_v) *is continuous and strictly increasing in r and in the radii r_i that are finite.*

Proof. The claim follows straight from the previous lemma and from the fact that the angle sum $\theta_R(v)$ at a vertex v is the sum of the central angles of the faces of the flower of v. ☐

Corollary 4.13. *Let $F_v = \{v; v_1, ..., v_n\}$ be the combinatorial closed flower of an interior vertex v and let $\{r_1, ..., r_n\}$ be a hyperbolic label for its neighbours. Then there exists an unique label r^* for v such that, if F_v is equipped with the label $R = \{r^*; r_1, ..., r_n\}$, then $\theta_R(v) = 2\pi$.*

Proof. Let r be a finite label for v. For the auxiliary functions g_i used in the proof of the Lemma 4.11 holds

$$\lim_{r_1 \downarrow 0} g_i(x(r_1)) = \lim_{x \downarrow 1} g_i(x) = \pi \quad \text{and} \quad \lim_{r_1 \to \infty} g_i(x(r_1)) = \lim_{x \to \infty} g_i(x) = 0$$

for $i = 1, 2, 3$. Hence, by continuity and monotonicity of the angle sum map, the angle sum at v, when considered as a function of r, is a bijection to the interval $[0, n\pi)$. Since F_v is closed, the number n of the neighbours of v has to be at least three. ☐

Corollary 4.14. *Let $F_v = \{v; v_1, ..., v_n\}$ be a combinatorial closed flower of an interior vertex v and let $R = \{r; r_1, ..., r_n\}$ be a hyperbolic label for F_v. Suppose that the angle sum at v satisfies $\theta(v) \geq 2\pi$. Then $r \leq \log(\sin(\frac{\pi}{n}))$.*

Proof. Since the angle sum $\theta_R(v)$ at v is strictly decreasing in r and strictly increasing in the radii of the neighbours of v, it is sufficient to study the situation where all the radii of the neighbours are infinite and r is such a real number that $\theta_R(v) = 2\pi$. In this extreme case, the central angle α at v is by Lemma 4.10 common to all the faces of v. Since the packing condition is satisfied at v, the corresponding geometrical flower F_{c_v} for F_v exists and hence $\alpha = 2\pi/n$. By Equation 4.1

$$\arccos(1 - 2e^{-2r}) = \frac{2\pi}{n},$$

from which follows that

$$\begin{aligned} e^{-2r} &= \frac{1}{2}(1 - \cos(\frac{2\pi}{n})) \\ &= \sin^2(\frac{\pi}{n}) \end{aligned}$$

and further
$$r = -\log(\sin(\frac{\pi}{n})).$$
$\square$

4.2.2 Proof of the existence of the maximal packing

Theorem 4.15 (Existence of the maximal packing)**.** *Let $\mathscr{T}$ be a triangulation of the closed disk. Let us assume that there exists a circle packing P_0 for $\mathscr{T}$ in $\mathbb{D}$. Then there exists an essentially (up to the automorphisms of $\mathbb{D}$) unique univalent circle packing $\mathcal{P}_{\mathscr{T}}$ for $\mathscr{T}$ in $\mathbb{D}$ such that every boundary circle is a horocycle. We shall call $\mathcal{P}_{\mathscr{T}}$ the maximal packing for $\mathscr{T}$.*

Proof. We shall divide the proof in several lemmas. Let us define a collection Φ of labels by
$$\Phi = \{R : \theta_R(v) \geq 2\pi \text{ for every } v \in \mathscr{T}_{int}\}$$
and the supremum $\tilde{R}$ of Φ by
$$\tilde{R}(v) = \sup_{R\in\Phi}\{R(v)\} \quad \text{for every } v \in \mathscr{T}.$$
First we shall check couple of properties of Φ that are needed later on, as we proceed with the proof of the theorem.

Lemma 4.16. *Φ is non-empty and closed under taking maximum.*

Proof. By the assumption of the theorem, there exists a circle packing P_0 for $\mathscr{K}$, whose radii form a label R_0 belonging to Φ. Hence Φ is non-empty.
Let us assume that $R_1, R_2 \in \Phi$ and $R = \max\{R_1, R_2\}$. Let us choose an arbitrary interior vertex $v \in \mathscr{T}_{int}$ and suppose without loss of generality that $R(v) = R_1(v)$. The labels for the neighbours of v in R are greater or equal as the corresponding labels in R_1. By the monotonicity in flowers (Corollary 4.12)
$$\theta_R(v) \geq \theta_{R_1}(v) \geq 2\pi$$
and hence $R \in \Phi$. $\square$

Our ultimate goal is to prove that $\tilde{R}$ is a packing label for $\mathscr{T}$ providing us with the maximal packing $\mathcal{P}_{\mathscr{T}}$ of the theorem. Next we shall show that $\tilde{R}$ possesses the necessary properties for that to happen.

Lemma 4.17. *$\tilde{R}(v) < \infty$ for every $v \in \mathscr{T}_{int}$ and $\tilde{R}(w) = \infty$ for every $w \in \mathscr{T}_{bnd}$.*

Proof. Let us choose an arbitrary interior vertex v and assume that n is the degree of v. Since $\theta_R(v) \geq 2\pi$ for every $R \in \Phi$, from Corollary 4.14 follows that $R(v) \leq -\log(\sin(\frac{\pi}{n}))$ for every $R \in \Phi$. Hence $\tilde{R}(v) \leq -\log(\sin(\frac{\pi}{n}))$ completing the proof of the first claim of the lemma. Let us assume now that $w \in \mathscr{T}_{bnd}$ and R is a label in Φ. By setting $R(w) = \infty$, the angle sums at the interior vertices can only increase according to Lemma 4.12. Hence the modified label belongs to Φ. It follows that $\tilde{R}(w) = \infty$ for every $w \in \mathscr{T}_{bnd}$. □

Lemma 4.18. *$\tilde{R}$ is a packing label for $\mathscr{T}$.*

Proof. By the previous lemma and the definition of $\tilde{R}$, for every interior vertex $v \in \mathscr{T}_{int}$ and strictly positive integer i there exists $R_{v,i} \in \Phi$ such that

$$|\tilde{R}(v) - R_{v,i}(v)| < \frac{1}{i}.$$

By Lemma 4.16

$$R_i := \max_v\{R_{v,i}\} \in \Phi \quad \text{for every } i \in \mathbb{N}.$$

The label R_i satisfies

$$|\tilde{R}(v) - R_i(v)| < \frac{1}{i} \quad \text{for every } v \in \mathscr{T}_{int}.$$

We may also assume that $R_i(v) = \infty$ for every $w \in \mathscr{T}_{bnd}$ and therefore

$$\lim_{i\to\infty} R_i(v) = \tilde{R}(v) \quad \text{for every } v \in \mathscr{T}.$$

By continuity of the angle sum (Lemma 4.12) $\theta_{R_i}(v) \to \theta_{\tilde{R}}(v)$ as $i \to \infty$ and hence $\theta_{\tilde{R}}(v) \geq 2\pi$ for every interior vertex v. Let us now assume that $\theta_{\tilde{R}}(v) > 2\pi$ for some interior vertex v. Then it would be possible to increase the label of v in such a way that the angle sum at v would still be at least 2π. By doing so, the angle sums of the neighbouring vertices could only increase, implying that the modified label would belong to Φ. This would contradict the definition of $\tilde{R}$. Therefore $\tilde{R}(v) = 2\pi$ for every interior vertex v and $\tilde{R}$ is an unbranched packing label for $\mathscr{T}$.

□

Since $\tilde{R}$ is a packing label for $\mathscr{T}$, Theorem 4.7 guarantees that there exists a circle packing $\mathcal{P}_{\mathscr{T}}$ with $\mathcal{P}_{\mathscr{T}} \longleftrightarrow \mathscr{T}(\tilde{R})$ implying that the boundary circles of $\mathcal{P}_{\mathscr{T}}$ are horocycles. We have still left to prove that $\mathcal{P}_{\mathscr{T}}$ is essentially unique and univalent.

For the proof of univalence, we shall first give definitions of proper maps and covering maps and then state a pair of propositions, which we are not going to prove in this connection. However, a proof for the first proposition can be studied from Bredon's book [4]. Information about covering maps and a proof for the second proposition can be found from Lee's book [7].

Definition 4.19 (Proper maps)**.** Let $f : X \to Y$ be a map between two topological spaces. If $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$, then f is a proper map.

Definition 4.20 (Covering maps)**.** Let X be a path connected and let Y be a locally path connected topological space. Let $f : X \to Y$ be a continuous and surjective map. Assume that every point $q \in Y$ has a connected and open neighbourhood U such that each component of $f^{-1}(U) \subset X$ is mapped homeomorphically onto U by f. Then f is a covering map.

Proposition 4.21 (Invariance of domain)**.** *Let M^n and N^n be topological n-manifolds and let $f : M^n \to N^n$ be a continuous bijection. Then f is a homeomorphism.*

Proposition 4.22. *Let X and Y be two topological spaces and let Y be simply connected. If $\phi : X \to Y$ is a covering map, then ϕ is a homeomorphism.*

Lemma 4.23. *The maximal packing $\mathcal{P}_{\mathscr{T}}$ is univalent.*

Proof. Since $\tilde{R}$ is an unbranched packing label for $\mathscr{T}$, the corresponding circle packing $\mathcal{P}_{\mathscr{T}}$ is locally univalent.
By Definitions 4.1 and 4.3 we may argue that there exists a continuous map $\phi : \mathscr{T} \to \mathrm{carr}(\mathcal{P}_{\mathscr{T}})$ mapping each face of $\mathscr{T}$ bijectively to the corresponding triangle in $\mathrm{carr}(\mathcal{P}_{\mathscr{T}})$. The map ϕ is clearly locally bijective at the interior points of the faces of $\mathscr{T}$. Again by Definition 4.1, we may argue that ϕ is locally bijective at the interior points of the edges of $\mathscr{T}$. The packing condition $\tilde{R}(v) = 2\pi$ at the interior vertices of $\mathscr{T}$ guarantees that ϕ is locally bijective at the interior vertices of $\mathscr{T}$. Since every boundary circle of $\mathcal{P}_{\mathscr{T}}$ is a horocycle, the circles of the flower of a boundary circle are non-overlapping. Therefore ϕ is locally bijective also at the boundary vertices of $\mathscr{T}$.
Next we shall modify ϕ on the boundary faces of $\mathscr{T}$ in such a way that the image of $\mathscr{T}$ coincides with the closed disk $\bar{\mathbb{D}}$. This is done by pushing the images of the boundary edges of $\mathscr{T}$ continuously to the boundary of $\bar{\mathbb{D}}$ at the same time leaving ϕ unchanged at the vertices and at the interior edges of $\mathscr{T}$. We are also going to preserve continuity and local bijectivity of ϕ in the modification.

To prove that ϕ is a proper map, let us assume that $K \subset \bar{\mathbb{D}}$ is compact. Then K is also closed. By continuity of ϕ, the inverse image $\phi^{-1}(K)$ of K under ϕ is closed. Since a closed subset of a compact space is compact, we conclude that ϕ is a proper map.
To put together, ϕ is a continuous, surjective, locally injective and proper map from $\mathscr{T}$ to $\bar{\mathbb{D}}$. Next we shall prove that ϕ is a covering map. Let $q \in \bar{\mathbb{D}}$ be an arbitrary point. Suppose that the set $\phi^{-1}(q) \subset \mathscr{T}$ is infinite, which would imply that it has a limit point p in $\mathscr{T}$. If $p \in \phi^{-1}(q)$, then ϕ would not be locally injective at p. If $p \notin \phi^{-1}(q)$, then the preimage of q under ϕ would not be a compact set, which would contradict properness of ϕ. Therefore $\phi^{-1}(q)$ has to be a finite set. Hence we can find such an open neighbourhood U of q that each component of $\phi^{-1}(U)$ is bijectively and continuously mapped onto U by ϕ.
By applying Proposition 4.21 to the components of $\phi^{-1}(U)$ we obtain that ϕ is a covering map and further, according to Proposition 4.22, ϕ is a homeomorphism. The packing $\mathcal{P}_{\mathscr{T}}$ is covered by the images of the faces of $\mathscr{T}$ under ϕ. Hence by bijectivity of ϕ, we may conclude that $\mathcal{P}_{\mathscr{T}}$ is an univalent circle packing for $\mathscr{T}$. □

Lemma 4.24. *The maximal packing $\mathcal{P}_{\mathscr{T}}$ is essentially unique.*

Proof. By Lemma 4.10, the area of a hyperbolic triangle in the carrier of $\mathcal{P}_{\mathscr{T}}$ is equal to $\pi - \alpha_1 - \alpha_2 - \alpha_3$. Hence

$$\text{area}(\text{carr}(\mathcal{P}_{\mathscr{T}})) = F\pi - \sum$$

where F is the number of faces of $\mathscr{T}$ and $\sum$ is the sum of the angles in the triangles of carr($\mathcal{P}_{\mathscr{T}}$). By considering separately the angles occurring at exterior vertices and the angles occurring at interior vertices, we may write

$$\text{area}(\text{carr}(\mathcal{P}_{\mathscr{T}})) = F\pi - \sum_{int} - \sum_{bnd}$$

where $\sum_{bnd} = 0$, since the boundary circles of $\mathcal{P}_{\mathscr{T}}$ are horocycles. Whereas if n denotes the number of interior vertices of $\mathscr{T}$, then $\sum_{int} = 2\pi n$. Thus

$$\text{area}(\text{carr}(\mathcal{P}_{\mathscr{T}})) = \pi(F - 2n).$$

Let us now assume that $\mathscr{T}$ has an another unbranched packing label R' associating infinite labels with the boundary vertices of $\mathscr{T}$. Let us also assume that P' is a circle packing for $\mathscr{T}$ with $P' \longleftrightarrow \mathscr{T}(R')$. From the equation above, it can be seen that the carrier of P' has the same area as the carrier of $\mathcal{P}_{\mathscr{T}}$. Since R' belongs to Φ, it holds that $R'(v) \leq \tilde{R}(v)$ for every

$v \in \mathscr{T}$. If the inequality would be strict even at a single vertex, then Lemma 4.12 would imply

$$\text{area}(\text{carr}(P')) < \text{area}(\text{carr}(\mathcal{P}_{\mathscr{T}})).$$

Hence $R'(v) = \tilde{R}(v)$ for every $v \in \mathscr{T}$. Therefore $\tilde{R}$ is unique and consequently $\mathcal{P}_{\mathscr{T}}$ is unique up to the isometries of $\mathbb{D}$. That is, $\mathcal{P}_{\mathscr{T}}$ is essentially unique. □

By Lemmas 4.18, 4.23 and 4.24, we may conclude that the claim of Theorem 4.15 holds.

□

4.2.3 Existence of a circle packing in the hyperbolic disk

Theorem 4.25. *Let $\mathscr{T}$ be a triangulation of the closed disk. Then there exists an essentially (up to the automorphisms of $\mathbb{D}$) unique univalent circle packing $\mathcal{P}_{\mathscr{T}}$ for $\mathscr{T}$ in $\mathbb{D}$ such that every boundary circle is a horocycle.*

Proof. By Theorem 4.15, it is sufficient to show that there exits a circle packing for $\mathscr{T}$. Our plan is to use induction in the total number V of vertices of $\mathscr{T}$. The minimal case occurs when $\mathscr{T}$ is the trivial triangulation and hence $V = 3$. In this case, the maximal packing $\mathcal{P}_{\mathscr{T}}$ of $\mathscr{T}$ is a triple with all the circles as horocycles.
Let us now assume that $V > 3$ and that the induction hypothesis holds for complexes having less than V vertices. Let us choose an arbitrary boundary vertex $w \in \mathscr{T}_{bnd}$ incident with at least one of the interior edges of $\mathscr{T}$. Since $V > 3$, such a vertex exists. We shall examine two different possibilities separately.

Case 1: There exists $\langle wu \rangle \in \mathscr{T}_{int}$ such that $u \in \mathscr{T}_{bnd}$.

We shall give the edge $\langle wu \rangle$ an orientation directed from w to u. Next we shall divide $\mathscr{T}$ into two pieces $\mathscr{T}_1$ and $\mathscr{T}_2$ along the edge $\langle wu \rangle$ by duplicating the vertices w and u, and the edge $\langle wu \rangle$. We may choose $\mathscr{T}_1$ and $\mathscr{T}_2$ to be the pieces on the right side and on the left side of the edge $\langle wu \rangle^*$ respectively. The 2-complexes $\mathscr{T}_1$ and $\mathscr{T}_2$ are triangulations of the closed disk having at most $V - 1$ vertices. This division is illustrated in Figure 4.5.
By the induction hypothesis and Theorem 4.15, there exists the maximal packings $\mathcal{P}_1$ and $\mathcal{P}_2$ for $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively, with all boundary circles as horocycles.

Let us first consider the maximal packing $\mathcal{P}_1$. There exists an automorphism Φ_1 of the hyperbolic disk that maps the circles of $\mathcal{P}_1$ for the vertices w and u to the circles

$$c_w = \left\{ \left| z - \frac{1}{2} \right| = \frac{1}{2} \right\} \quad \text{and} \quad c_u = \left\{ \left| z + \frac{1}{2} \right| = \frac{1}{2} \right\} \tag{4.5}$$

respectively. The choice of the orientation implies that the rest of the circles of the packing $\mathcal{P}_1$ lie now above the real axis. In practice, the mapping of the packing $\mathcal{P}_1$ by this automorphism can be realized by starting to lay down the circles similarly as we did in the proof of Theorem 4.7. More precisely, with the maximal packing $\mathcal{P}_1$ comes also a packing label R_1 for $\mathscr{T}_1$. After placing the circles c_w and c_u as suggested above, add a third circle with the radii given by the label R_1 externally tangent to c_w and c_u following the combinatorics of $\mathscr{T}_1$ and the given orientation. A triangle for a base face is formed and the rest of the packing can be obtained through developments along chains starting from the base face.

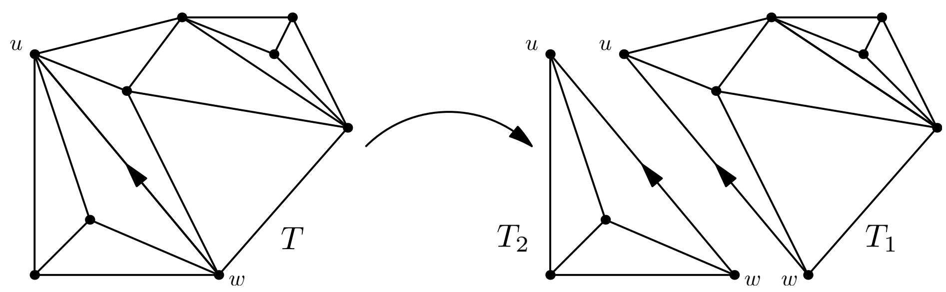


Figure 4.5: An example of the division of Case 1.

Similarly, there exists an automorphism Φ_2 that maps the circles of $\mathcal{P}_2$ for the vertices w and u to the circles c_w and c_u respectively. Now the choice of the orientation implies that the rest of the circles of the packing $\mathcal{P}_2$ lie below the real axis. Let P_1 and P_2 be the images of the maximal packings $\mathcal{P}_1$ and $\mathcal{P}_2$ under the automorphisms Φ_1 and Φ_2 respectively. We shall glue P_1 and P_2 together by superimposing the circles c_w and c_u resulting an univalent circle packing for $\mathscr{T}$ in $\mathbb{D}$.

Case 2: $u \in \mathscr{T}_{int}$ for every $\langle wu \rangle \in \mathscr{T}_{int}$.

First we shall erase vertex w and all the edges and faces incident with w from $\mathscr{T}$. Let us denote the remaining triangulation of the closed disk with $\mathscr{T}'$. Since $\mathscr{T}'$ has $V-1$ vertices, we may use the induction hypothesis to

obtain the maximal packing $\mathcal{P}'$ for $\mathscr{T}'$ in $\mathbb{D}$.
Next we shall move the packing $\mathcal{P}'$ from the hyperbolic disk into the complex plane and add two new circles to our configuration. The first of the added circles is the unit circle and it will serve as the circle for the vertex w. Therefore, let us denote this circle with c_w. The complement of the closed unit disk will be considered as the interior of c_w. Notice that c_w will be tangent with all the boundary circles of $\mathcal{P}'$ and hence it satisfies all the tangencies defined in $\mathscr{T}$. The tangencies of c_w that are not described in $\mathscr{T}$ will be considered as extraneous. Let us denote the union of the packing $\mathcal{P}'$ and the circle c_w with P''. The other circle we add shall be disjoint from the packing $\mathcal{P}'$ and the unit circle c_w, but it shall lie inside the unit disk. Let us denote this circle with C, its radius with r_C and its centre with z_C.
Next we shall use a Möbius transformation ϕ to obtain from P'' a circle packing for $\mathscr{T}$ in $\mathbb{D}$. We shall work here in the extended complex plane $\bar{\mathbb{C}}$. Let us define

$$\phi_1(z) = z - z_C,\ \phi_2(z) = \frac{z}{r_C},\ \phi_3(z) = \frac{1}{z} \quad \text{and} \quad \phi(z) = (\phi_3 \circ \phi_2 \circ \phi_1)(z).$$

The Möbius transformation ϕ is a composition of translation, dilatation and inversion and it works as follows: ϕ_1 translates the centre of the circle C to the origin. After that, ϕ_2 dilates the complex plane in such a way that the circle C coincides with the unit disk. Since C was disjoint from the other circles, it follows that the images of $\mathcal{P}'$ and c_w lie outside the closed unit disk. Finally, ϕ_3 maps the exterior of the unit disk to the interior of the unit disk and vice versa. Circles will be mapped to circles and the tangencies will be preserved. This implies that $\phi(P'')$, after reversion to the hyperbolic disk, will be a circle packing for $\mathscr{T}$ in $\mathbb{D}$. □

An example of the technique used in the proof of Case 2 is provided in Figure 4.7. The starting complex $\mathscr{T}$ of the example is presented in Figure 4.6. The vertex to be removed is labelled with w, the edges to be removed are drawn with dashed lines and the faces to be removed are coloured with grey. The maximal packing $\mathcal{P}'$ of $\mathscr{T}'$ is displayed in Subfigure 4.7a. The packing P'' together with the circle C is shown in Subfigure 4.7b. The circle C is drawn with red and the boundary of the circle c_w coinciding with the boundary of the unit circle is drawn with black. Subfigure 4.7c presents the situation after the translation and Subfigure 4.7d after the dilatation. The packing $\phi(P'')$ for $\mathscr{T}$ is presented in Subfigure 4.7e. Some of the circles are too small to be visible in this figure. Finally, the maximal packing for the original complex $\mathscr{T}$ is presented in Subfigure 4.7f.

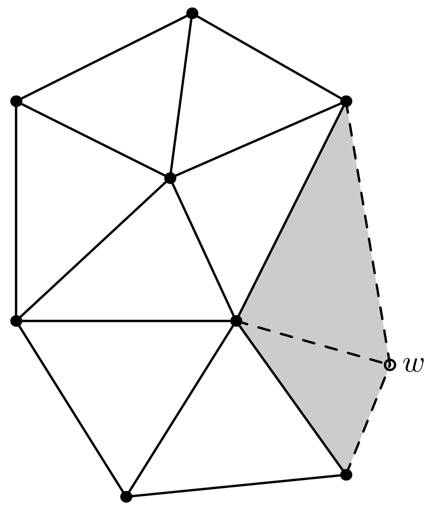


Figure 4.6: An example complex for Case 2.

4.3 Circle packings for uniform random triangulations

In Section 2.4 we described the principles of generating uniform random rooted triangulation of the disk. Theorem 4.25 guarantees the existence of the maximal packing for such a triangulation. The maximal packing is univalent and thus the carrier (Definition 4.3) of the packing provides us with an embedding of the triangulation in the hyperbolic disk. This section consists of collections of figures of embeddings of uniform random triangulations. In the figures, the starting vertex of the root edge of a triangulation is denoted by p_1 and the end vertex by p_2. If the root edge were not to be specified in a figure, the triangulation of the embedding would be unique only up to the isomorphisms, instead of the root-isomorphisms of triangulations.
All the figures are made by using Kenneth Stephenson's CirclePack program [10]. In Figure 4.8 is presented embeddings of random triangulations with 13 boundary vertices and varying number of interior vertices. In Figure 4.9 is presented embeddings of triangulations with 10 interior vertices and varying number of boundary vertices. In Figure 4.10 is illustrated the situation, where the number of interior vertices is small in relation to the number of boundary vertices and vice versa. Figure 4.11 consists of a few more examples of embeddings of random triangulations.

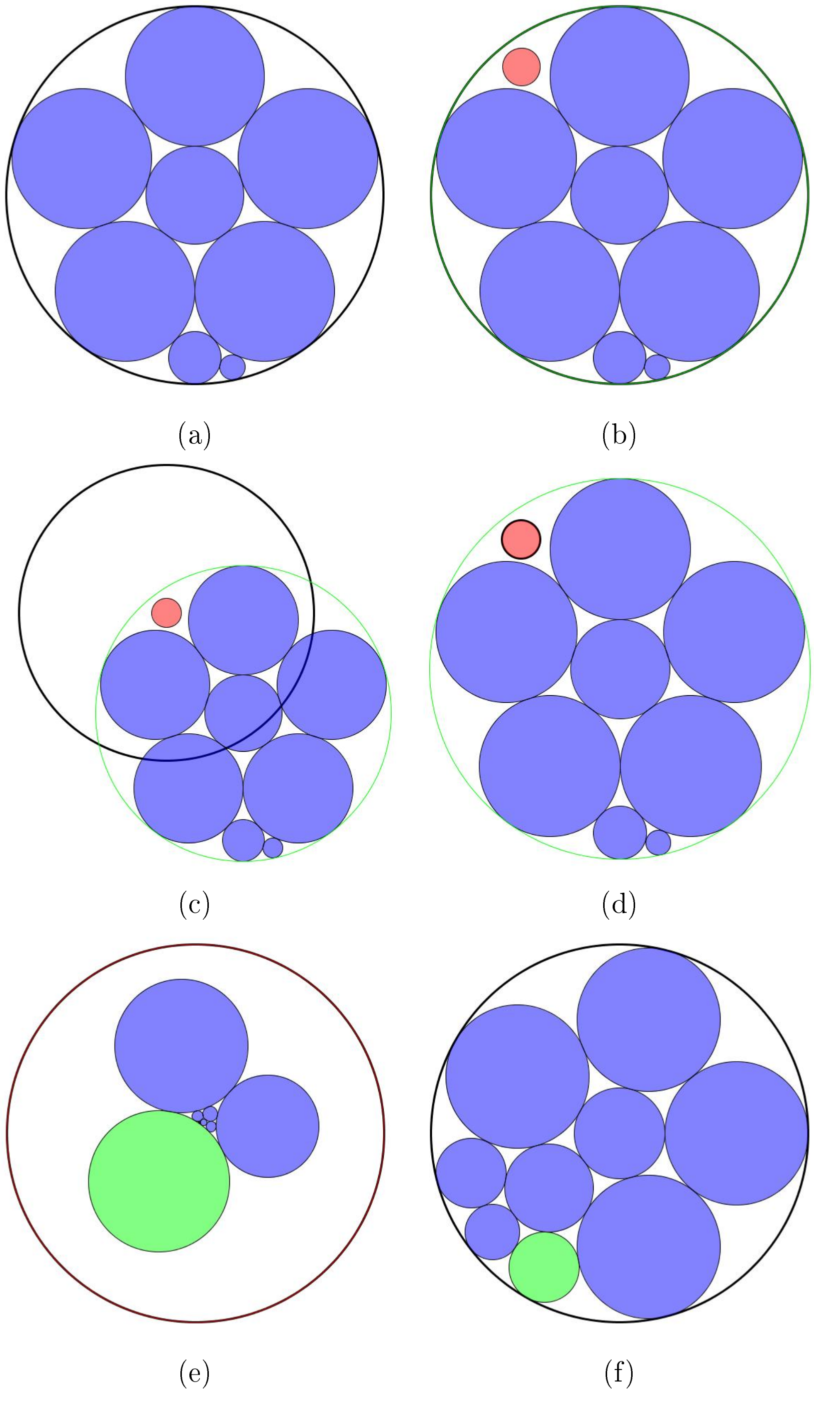
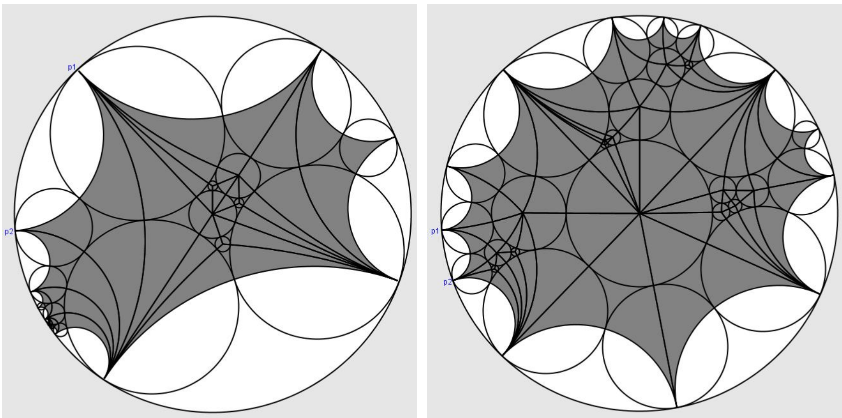
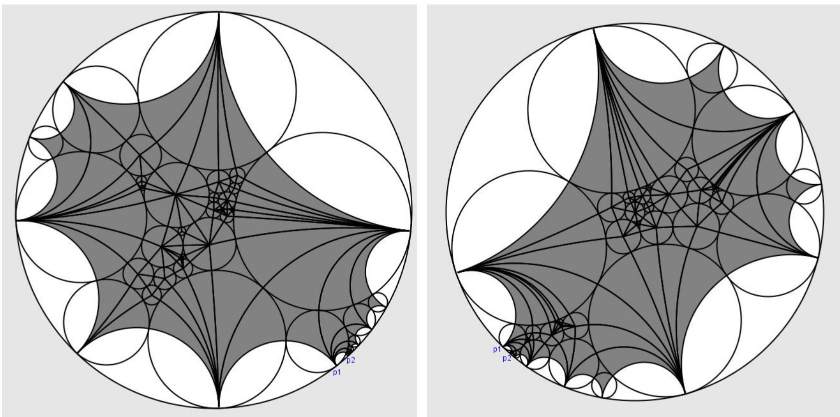


Figure 4.7: An example of Case 2

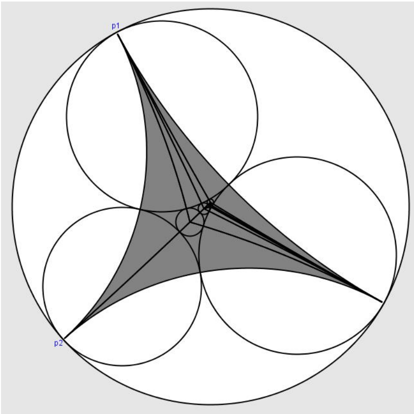


(a) $n = 5, m = 10$ (b) $n = 15, m = 10$

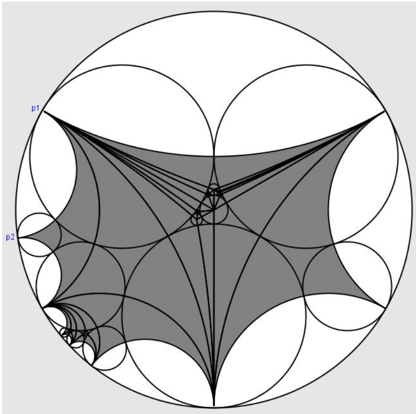


(c) $n = 15, m = 10$ (d) $n = 34, m = 10$

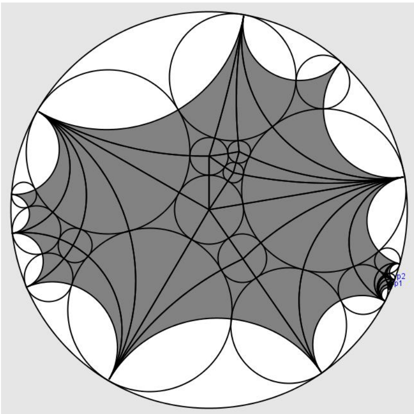
Figure 4.8: Maximal circle packings of random triangulations with fixed boundary length.



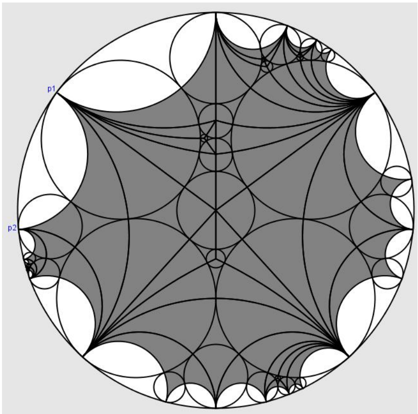
(a) $n = 10, m = 0$



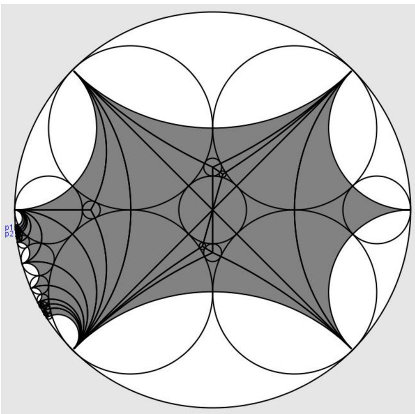
(b) $n = 10, m = 6$



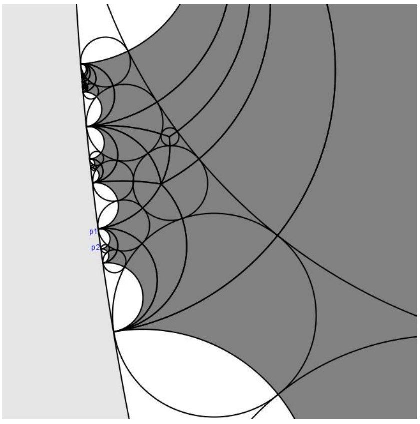
(c) $n = 10, m = 12$



(d) $n = 10, m = 18$

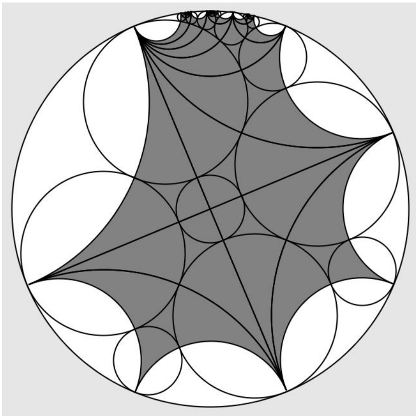


(e) $n = 10, m = 30$

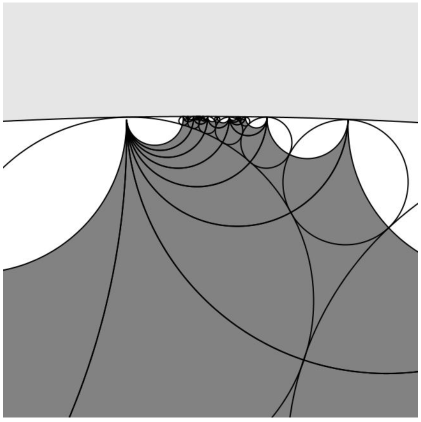


(f) 4.9e zoomed in.

Figure 4.9: Maximal circle packings of random triangulations with fixed number of interior vertices.

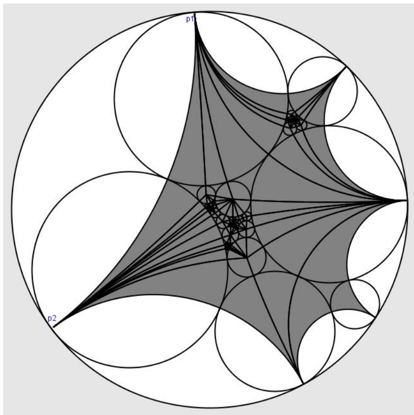


(a) $n = 1, m = 47$

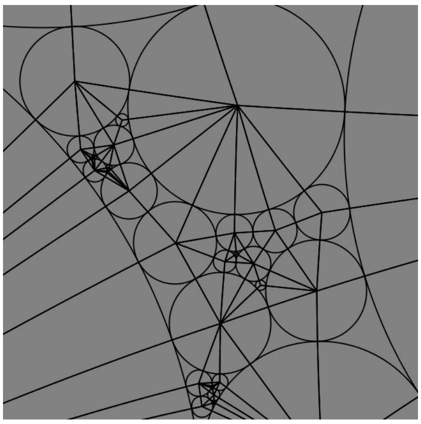


(b) 4.10a zoomed in.

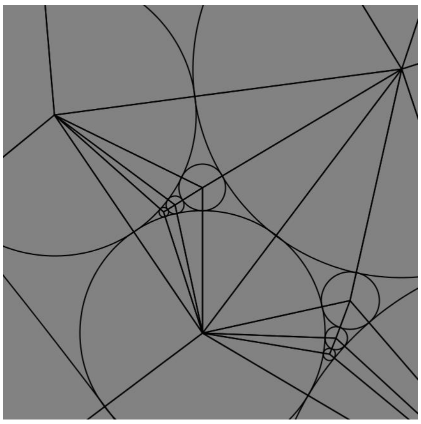
(c) 4.10b zoomed in.



(d) $n = 40, m = 3$

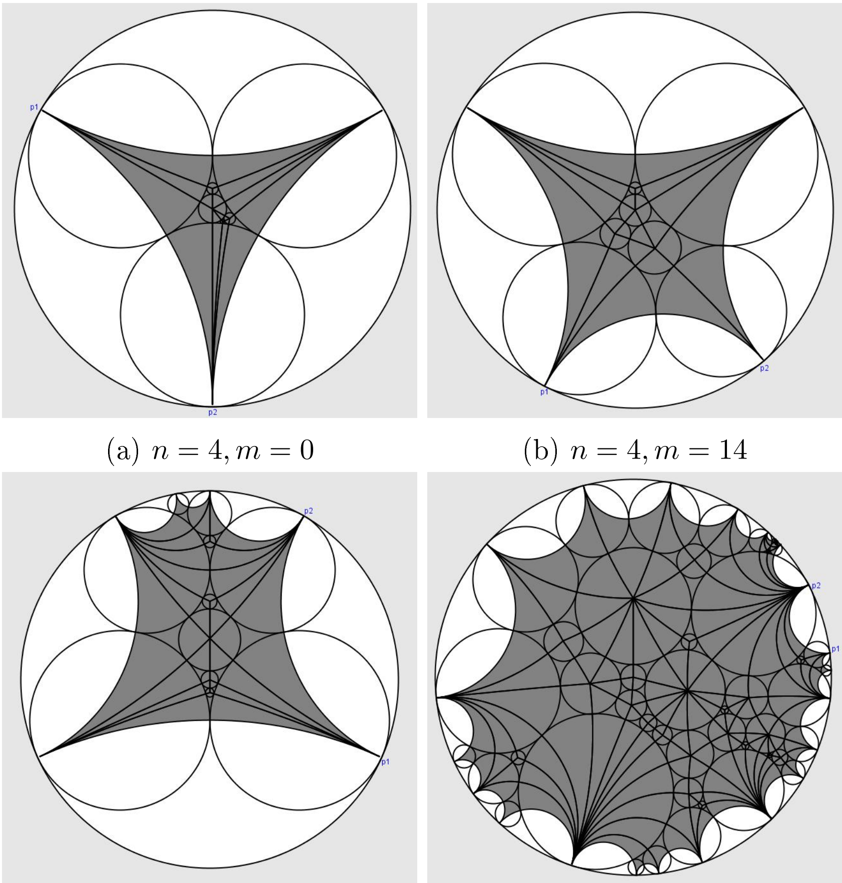


(e) 4.10d zoomed in.



(f) 4.10e zoomed in.

Figure 4.10: Maximal circle packings of random triangulations and their zoomings.



(a) $n = 4, m = 0$

(b) $n = 4, m = 14$

(c) $n = 5, m = 3$

(d) $n = 25, m = 21$

Figure 4.11: Collection of circle packings of random triangulations.

Chapter 5

Distributions of random triangulations

The first section of this chapter concerns classes of triangulations with a fixed total number of vertices and the latter section classes of triangulations with a fixed boundary length. We shall begin both chapters by defining probability measures on the classes at hand. After that, we shall define random variables that are mappings from the formed probability spaces to natural numbers. The random variables of the first section return the boundary length of a triangulation and the random variables of the latter section return an appropriately scaled number of vertices of a triangulation. The main results of the chapter are Theorems 5.4 and 5.12 providing us with convergence in distribution of these random variables. The statements of the theorems are presented in the early parts of the sections, whereas the proofs are provided only after we have gathered all the necessary tools for them.
The basic idea of the proof of Lemma 5.6 is suggested in the paper [3] by Angel and Schramm. Also some motivation for the definition of the probability measures of the second section can be found from the same paper. The need for this kind of convergence results that we are about to derive arise for example when attempting to discretize path integrals related to 2D quantum gravity.
However, before proceeding any further into the subject of randomness in triangulations we shall state Stirling's approximation, which shall be used numerous times when we are estimating the factorials occurring in the formula of the number of triangulations (Equation 3.13).

Proposition 5.1 (Stirling's approximation)**.**

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)$$

implying that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1.$$

Let $l = m + 3$ denote the number of boundary vertices and $V = n + l$ the total number of vertices of a triangulation. Let us define a function $D(V, l)$ returning the number of isomorphism classes of triangulations of type $[V - l, l - 3]$. By Theorem 3.6

$$D(V, l) = \frac{2(2l-3)!(4V-2l-5)!}{(l-3)!(l-1)!(V-l)!(3V-l-3)!} \tag{5.1}$$

when $V, l \in \mathbb{N}$, $l \geq 3$ and $V \geq l$. Otherwise let us set $D(V, l) = 0$.

5.1 Distribution of a random boundary length

Let us fix V and define a class

$$\mathcal{T}_{(V)} = \bigcup_{\substack{n,m\in\mathbb{N}\cup\{0\} \\ n+m+3=V}} \mathcal{T}_{n,m}$$

of triangulations having V vertices in total.

Definition 5.2 (Probability measure on $\mathcal{T}_{(V)}$)**.** Since $\mathcal{T}_{(V)}$ is a finite set, we may define an uniform probability measure μ_V on $\mathcal{T}_{(V)}$ by

$$\mu_V(A) = \frac{|A|}{|T_{(V)}|} \qquad \text{for every } A \subset \mathcal{T}_{(V)}.$$

Then $\left(\mathcal{T}_{(V)}, \mathcal{P}\left(\mathcal{T}_{(V)}\right), \mu_V\right)$ is a probability space, where the sigma-algebra of events is the power set $\mathcal{P}\left(\mathcal{T}_{(V)}\right)$ of $\mathcal{T}_{(V)}$.

Definition 5.3 (Random boundary length on $\mathcal{T}_{(V)}$)**.** Let us define a random variable $L_V : \mathcal{T}_{(V)} \to \mathbb{N}$ returning the boundary length l of a triangulation by

$$L_V(\mathscr{T}) = l \qquad \text{for every } \mathscr{T} \in \mathcal{T}_{(V)}.$$

Then under the probability measure $P = \mu_V$ holds

$$P(L_V = l) = \frac{D(V, l)}{\sum_{l'=3}^{V} D(V, l')}.$$

The statement of the main theorem of this section is given already here. The proof of the theorem can be found on page 60.

Theorem 5.4. *Let us consider the probability spaces* $\left(\mathcal{T}_{(V)}, \mathcal{P}(\mathcal{T}_{(V)}), \mu_V\right)$ *and the random variables* $L_V : \mathcal{T}_{(V)} \longrightarrow \mathbb{N}$. *There exists a non-degenerate random variable* L *such that* L_V *converges in distribution to* L *as* $V \to \infty$.

Before proving the actual theorem, we shall state couple of lemmas that we shall need in our proof.

Lemma 5.5.

$$\lim_{V\to\infty} \frac{D(V,l)}{D(V,3)} = \frac{3^{l-4}(2l-3)!}{16^{l-3}(l-3)!(l-1)!} =: D_l(l)$$

Proof. By using Stirling's approximation (Proposition 5.1) to the factorials of D that involve V we obtain

$$\begin{aligned}
\lim_{V\to\infty} \frac{D(V,l)}{D(V,3)} &= \lim_{V\to\infty} \frac{(2l-3)!(4V-2l-5)!(V-3)!(3V-6)!}{3(l-3)!(l-1)!(V-l)!(3V-l-3)!(4V-11)!} \\
&= \frac{(2l-3)!}{3(l-3)!(l-1)!} \cdot \\
&\quad \lim_{V\to\infty} \frac{(4V-2l-5)^{4V-2l-5+\frac{1}{2}}(V-3)^{V-3+\frac{1}{2}}(3V-6)^{3V-6+\frac{1}{2}}}{(V-l)^{V-l+\frac{1}{2}}(3V-l-3)^{3V-l-3+\frac{1}{2}}(4V-11)^{4V-11+\frac{1}{2}}},
\end{aligned} \tag{5.2}$$

where the square root terms and the exponentials occurring in Stirling's approximation got cancelled out. Next we shall use the fact that if a and b are strictly positive real numbers then

$$\begin{aligned}
1 &= \lim_{V\to\infty} \frac{(aV-b)^{aV-b+\frac{1}{2}}}{(aV)^{aV-b+\frac{1}{2}}\left(1-\frac{b}{aV}\right)^{aV-b+\frac{1}{2}}} \\
&= \lim_{V\to\infty} \frac{(aV-b)^{aV-b+\frac{1}{2}}}{e^{-b}(aV)^{aV-b+\frac{1}{2}}}.
\end{aligned}$$

We shall apply this result to Equation 5.2 by making substitutions of type

$$(aV-b)^{aV-b+\frac{1}{2}} \to e^{-b}(aV)^{aV-b+\frac{1}{2}}.$$

The exponentials and the exponents of V will cancel out leaving

$$\begin{aligned}
\lim_{V\to\infty} \frac{D(V,l)}{D(V,3)} &= \frac{(2l-3)!}{3(l-3)!(l-1)!} \lim_{V\to\infty} \frac{4^{4V-2l-5+\frac{1}{2}}3^{3V-6+\frac{1}{2}}}{3^{3V-l-3+\frac{1}{2}}4^{4V-11+\frac{1}{2}}} \\
&= \frac{3^{l-4}(2l-3)!}{16^{l-3}(l-3)!(l-1)!}.
\end{aligned}$$

□

Lemma 5.6.

$$\lim_{V\to\infty}\sum_{l'=3}^{V}\frac{D(V,l')}{D(V,3)}=\sum_{l'=3}^{\infty}D_l(l')<\infty$$

Proof. Since $D(V,l')=0$ for $l'>V$, we may change the upper bound of the sum on LHS to infinity. By Equation 5.1

$$\frac{D(V,l'+1)}{D(V,l')}=\frac{(2l'-1)(2l'-2)(V-l')(3V-l'-3)}{l'(l'-2)(4V-2l'-5)(4V-2l-6)},$$

where

$$\begin{aligned}\frac{(2l'-1)(2l'-2)}{l'(l'-2)}&=\frac{\left(2-\frac{1}{l'}\right)\left(2-\frac{2}{l'}\right)}{1-\frac{2}{l'}}\\&=4+\mathcal{O}\left(\frac{1}{l'}\right).\end{aligned}\tag{5.3}$$

Let us set $0<r=\frac{l'}{V}\leq 1$ obtaining

$$\begin{aligned}\frac{(3V-l'-3)(V-l')}{(4V-2l'-5)(4V-2l'-6)}&=\frac{\left(3-\frac{l'}{V}-\frac{3}{V}\right)\left(1-\frac{l'}{V}\right)}{\left(4-\frac{2l'}{V}-\frac{5}{V}\right)\left(4-\frac{2l'}{V}-\frac{6}{V}\right)}\\&\leq\frac{(3-r)\,(1-r)}{(4-2r)^2+\mathcal{O}\left(\frac{1}{V}\right)}.\end{aligned}$$

Since $(4-2r)^2\in[4,16)$ we may expand the rational above about $(4-2r)^2$ obtaining

$$\begin{aligned}&(3-r)(1-r)\frac{1}{(4-2r)^2+\mathcal{O}\left(\frac{1}{V^2}\right)}\\&=(3-r)(1-r)\left(\frac{1}{(4-2r)^2}-\frac{\mathcal{O}\left(\frac{1}{V}\right)}{(4-2r)^4}+\frac{\mathcal{O}\left(\frac{1}{V^2}\right)}{(4-2r)^6}\cdots\right)\\&=\frac{(3-r)(1-r)}{(4-2r)^2}+\mathcal{O}\left(\frac{1}{V}\right)\\&=f(r)+\mathcal{O}\left(\frac{1}{V}\right).\end{aligned}$$

By derivating the auxiliary function $f(r)$ with respect to r we obtain

$$
\begin{aligned}
f'(r) &= \frac{(2r-4)(4-2r)+4(3-r)(1-r)}{(4-2r)^3} \\
&= \frac{-4}{(4-2r)^3},
\end{aligned}
$$

which is always negative. Since

$$f(0) = \frac{3}{16}$$

together with Equation 5.3 we obtain that

$$\frac{D(V,l'+1)}{D(V,l')} \leq \left(4+\mathcal{O}\left(\frac{1}{l'}\right)\right)\left(\frac{3}{16}+\mathcal{O}\left(\frac{1}{V}\right)\right).$$

That is, there exists $V^*, l^* \in \mathbb{N}$ such that when $V \geq V^*$ and $l' \geq l^*$ then

$$\frac{D(V,l'+1)}{D(V,l')} < \frac{4}{5}. \tag{5.4}$$

Let us consider the sum of the claim as an integral with respect to the counting measure in $\mathbb{N}$. By Equation 5.4 we have an integrable upper bound

$$\frac{D(V,l')}{D(V,3)} < \left(\frac{4}{5}\right)^{l'-l^*}\frac{D(V,l^*)}{D(V,3)}$$

for the terms of the sum when $l' \geq l^*$ and $V \geq V^*$. Therefore according to Lebesgue's dominated convergence theorem

$$
\begin{aligned}
\lim_{V\to\infty}\sum_{l'=3}^{\infty}\frac{D(V,l')}{D(V,3)} &= \sum_{l'=3}^{l^*-1}\lim_{V\to\infty}\frac{D(V,l')}{D(V,3)} + \lim_{V\to\infty}\sum_{l'=l^*}^{\infty}\frac{D(V,l')}{D(V,3)} \\
&= \sum_{l'=3}^{\infty}\lim_{V\to\infty}\frac{D(V,l')}{D(V,3)} \\
&= \sum_{l'=3}^{\infty} D_l(l') < \infty.
\end{aligned}
$$

□

Now we are ready to prove Theorem 5.4 by means of pointwise convergence of the cumulative distribution functions.

Proof of Theorem 5.4. Let $F_V(l)$ be the cumulative distribution function of L_V. By Lemma 5.6

$$
\begin{aligned}
\lim_{V\to\infty} P(L_V = l) &= \lim_{V\to\infty} \frac{D(V,l)}{\sum_{l'=3}^{V} D(V,l')} \\
&= \lim_{V\to\infty} \frac{D(V,l)}{\frac{D(V,3)}{D(V,3)} \sum_{l'=3}^{V} D(V,l')} \\
&= \frac{D_l(l)}{\sum_{l'=3}^{\infty} D_l(l')}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\lim_{V\to\infty} F_V(l) &= \lim_{V\to\infty} \sum_{l'=3}^{l} P(L_V = l') \\
&= \frac{\sum_{l'=3}^{l} D_l(l')}{\sum_{l'=3}^{\infty} D_l(l')}.
\end{aligned}
\tag{5.5}
$$

□

To be able to write the distribution of the limiting random variable explicitly we still need to determine the explicit value of the sum of the denominator of the expression above.

Lemma 5.7.

$$
\begin{aligned}
\sum_{l=3}^{\infty} D_l(l) &= \frac{1}{3} \left. \frac{(1-4x)^{\frac{3}{2}} + 6x - 1}{2x^2(1-4x)^{\frac{3}{2}}} \right|_{x=\frac{3}{16}} \\
&= \frac{256}{27}
\end{aligned}
$$

Proof. By changing the index of summation from l to $n = l - 3$, the sum of the statement can be written by the definition of $D_l(l)$ (Lemma 5.5) as

$$
\sum_{l=3}^{\infty} D_l(l) = \frac{1}{3} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{16}\right)^n (2n+3)!}{n!(n+2)!}.
$$

The rational function of the statement is analytic in the punctured disk $B(0, \frac{1}{4}) \setminus \{0\}$. By using L'Hospital's rule twice, we obtain that the limit of the function at the origin is equal to three. Hence by Riemann's removable singularity theorem, the function can be extended analytically to the disk $B(0, \frac{1}{4})$.

Our plan is to show that the Taylor expansion of the extended function about the origin is in fact

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{(2n+3)!}{(n+2)!},$$

which would conclude the proof of the lemma.

By expanding the square root of $(1-4x)$ about the origin we obtain

$$\frac{d^2}{dx^2}(1-4x)^{\frac{1}{2}} = \frac{d^2}{dx^2} \sum_{n=0}^{\infty} \frac{(2n)!}{(1-2n)(n!)^2} x^n$$

implying

$$(1-4x)^{-\frac{3}{2}} = \sum_{n=2}^{\infty} \frac{(2n)!n(n-1)}{4(2n-1)(n!)^2} x^{n-2}.$$

Now the rational function of the statement can be written as

$$\frac{(1-4x)^{\frac{3}{2}} + 6x - 1}{2x^2(1-4x)^{\frac{3}{2}}} = \frac{1}{2x^2} + \left(\frac{3}{x} - \frac{1}{2x^2}\right)\left(\sum_{n=2}^{\infty} \frac{(2n)!n(n-1)}{4(2n-1)(n!)^2} x^{n-2}\right),$$

where the terms involving negative powers of x will cancel each other out leaving

$$\begin{aligned} &\frac{3}{x}\left(\sum_{n=3}^{\infty} \frac{(2n)!n(n-1)}{4(2n-1)(n1)^2} x^{n-2}\right) - \frac{1}{2x^2}\left(\sum_{n=4}^{\infty} \frac{(2n)!n(n-1)}{8(2n-1)(n!)^2} x^{n-2}\right) \\ &= \sum_{k=0}^{\infty}\left(\frac{3\,(2(k+3))!(k+3)(k+2)}{4(2k+5)\left((k+3)!\right)^2} - \frac{(2(k+4))!(k+4)(k+3)}{4(2k+7)\left((k+4)!\right)^2}\right) x^k. \end{aligned} \tag{5.6}$$

Above the equality follows from changing the index of the summation of the first sum from n to $k = n-3$ and the index of the summation of the latter sum from n to $k = n-4$, and after that writing the two sums together. Let us denote the coefficient of x^k in the sum above with a_k. Then

$$\begin{aligned} a_k &= \frac{3(2k+6)!(k+3)(k+2)}{4(2k+5)\left((k+3)!\right)^2} - \frac{(2k+8))!(k+4)(k+3)}{8(2k+7)\left((k+4)!\right)^2} \\ &= \frac{6(2k+6)!(k+3)(k+2)(2k+7)(k+4) - (2k+8)!(k+3)(2k+5)}{8(2k+5)((k+3)!)^2(2k+7)(k+4)}, \end{aligned} \tag{5.7}$$

where the numerator can be written as

$$\begin{aligned}
&(2k+7)!(k+3)\big(6(k+2)(k+4)-(2k+8)(2k+5)\big)\\
&=(2k+7)!(k+3)\big(6(k^2+6k+8)-(4k^2+26k+40)\big)\\
&=(2k+7)!(k+3)(2k^2+10k+8)\\
&=2(2k+7)!(k+3)(k+4)(k+1).
\end{aligned}$$

By substituting in Equation 5.7 we obtain

$$\begin{aligned}
a_k &= \frac{(2k+6)!(k+3)(k+1)}{4(2k+5)((k+3)!)^2}\\
&= \frac{(2k+6)(2k+4)!(k+1)}{4(k+3)!(k+2)!}\\
&= \frac{(2k+6)(2k+3)!}{2(k+3)!k!}\\
&= \frac{(2k+3)!}{k!(k+2)!}
\end{aligned}$$

completing the proof of the lemma. □

By Equation 5.5 and Lemma 5.7, the random variables L_V converge in distribution to a random variable L, where the probability mass function $f_L(l)$ of L satisfies

$$f_L(l) = \begin{cases} \frac{27}{256}D_l(l), & l \in \{3,4,5,...\} \\ 0, & \text{otherwise.} \end{cases}$$

For the definition of $D_l(l)$ refer to statement of Lemma 5.5.
Values of the probability mass function $f_L(l)$ are presented in Figure 5.1 with linear scale and in Figure 5.2 with logarithmic scale. From Figure 5.2 can be seen that the probability decreases exponentially as the function of the boundary length. Values of the probability mass functions of the random variables L_{10}, L_{20} and L_{50} are presented in Figures 5.3, 5.4 and 5.5 respectively. The cumulative distribution function F_L of L is illustrated in Figure 5.6 and the cumulative distribution functions of L_{10}, L_{20} and L_{50} are illustrated in Figures 5.7, 5.8 and 5.9 respectively.

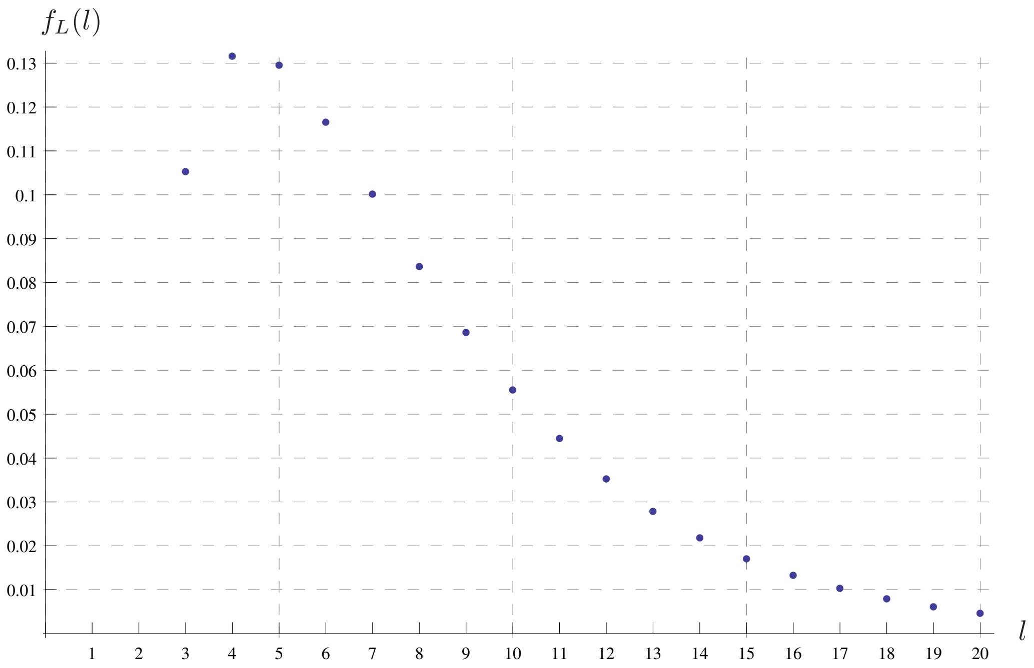


Figure 5.1: The probability mass function of L, linear scale.

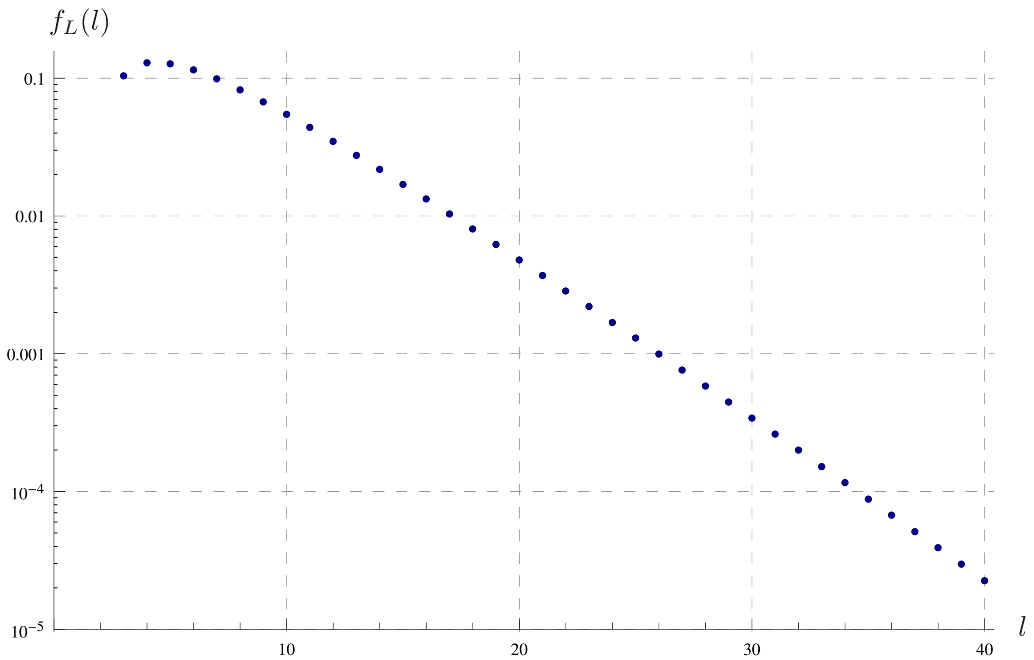


Figure 5.2: The probability mass function of L, logarithmic scale.

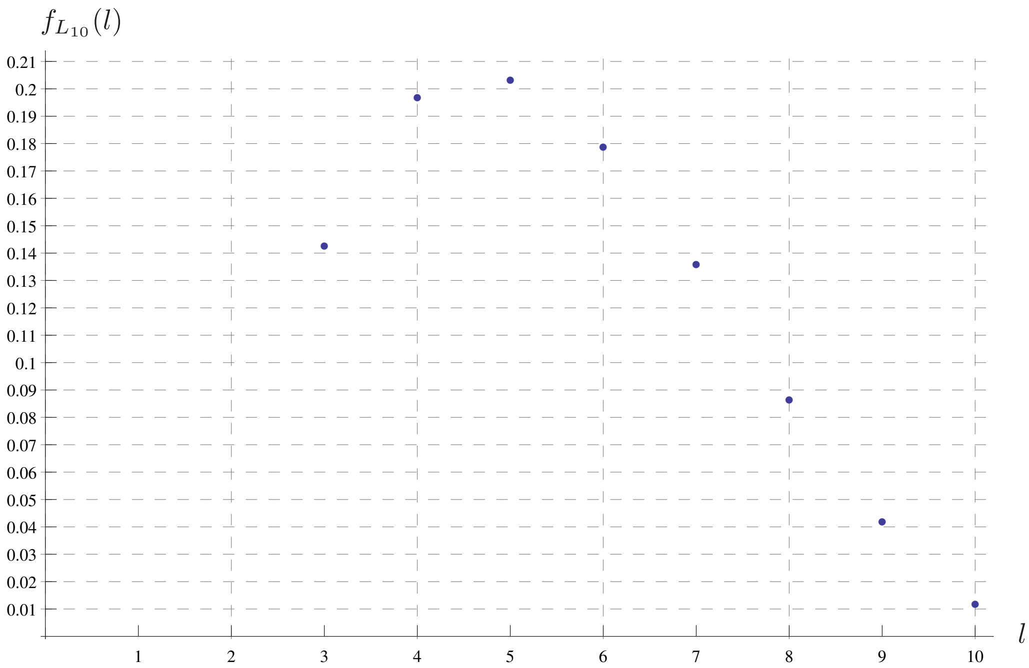


Figure 5.3: The probability mass function of L_{10}.

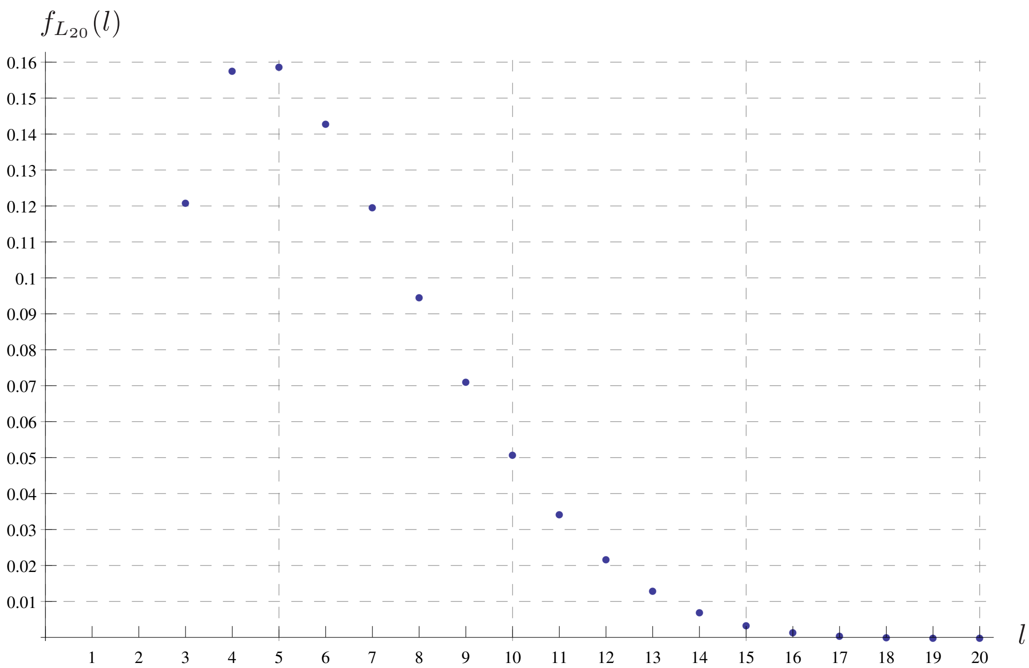


Figure 5.4: The probability mass function of L_{20}.

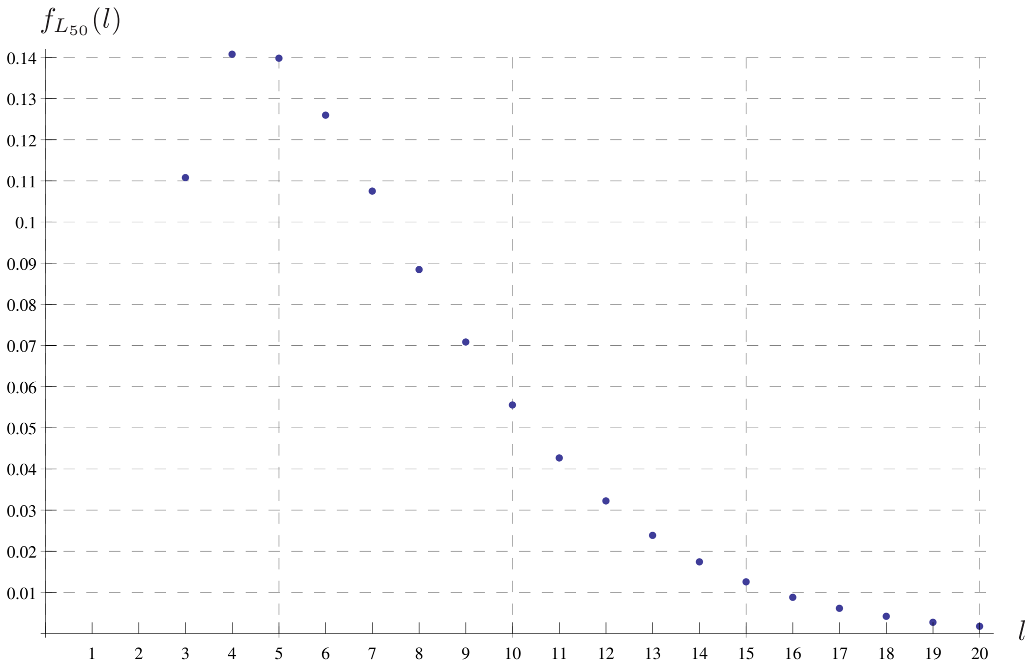


Figure 5.5: The probability mass function of L_{50}.

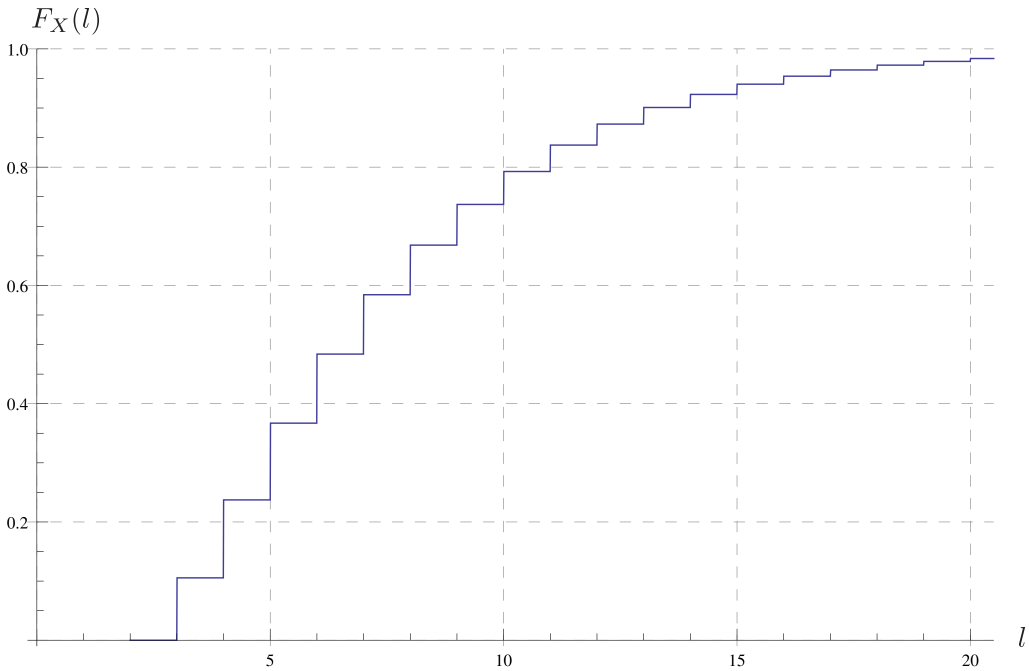


Figure 5.6: The cumulative distribution function of L.

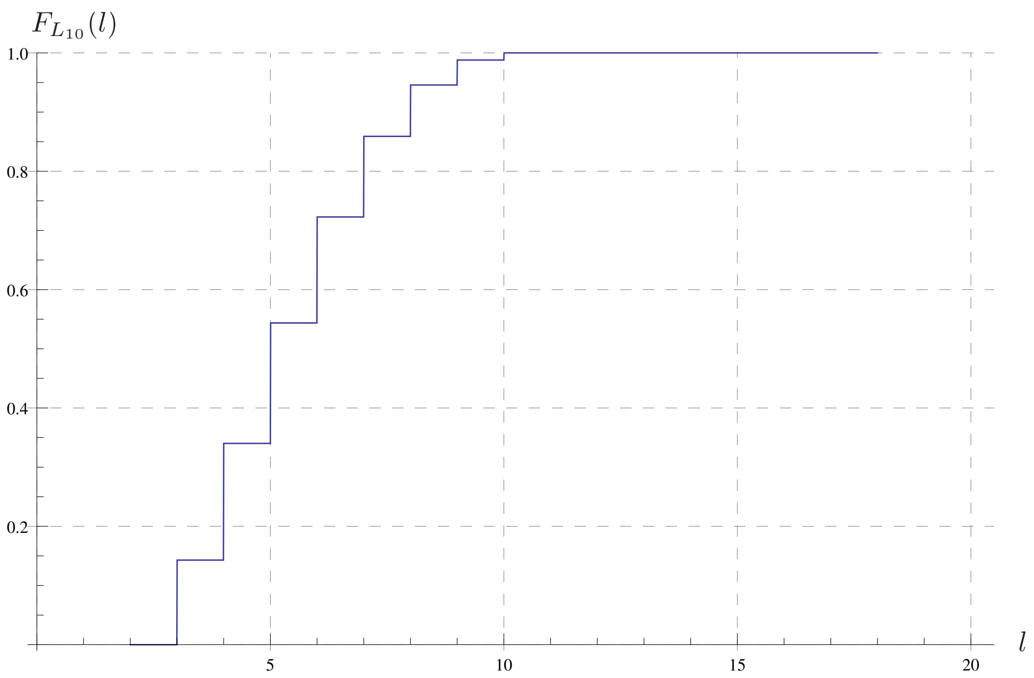


Figure 5.7: The cumulative distribution function of L_{10}.

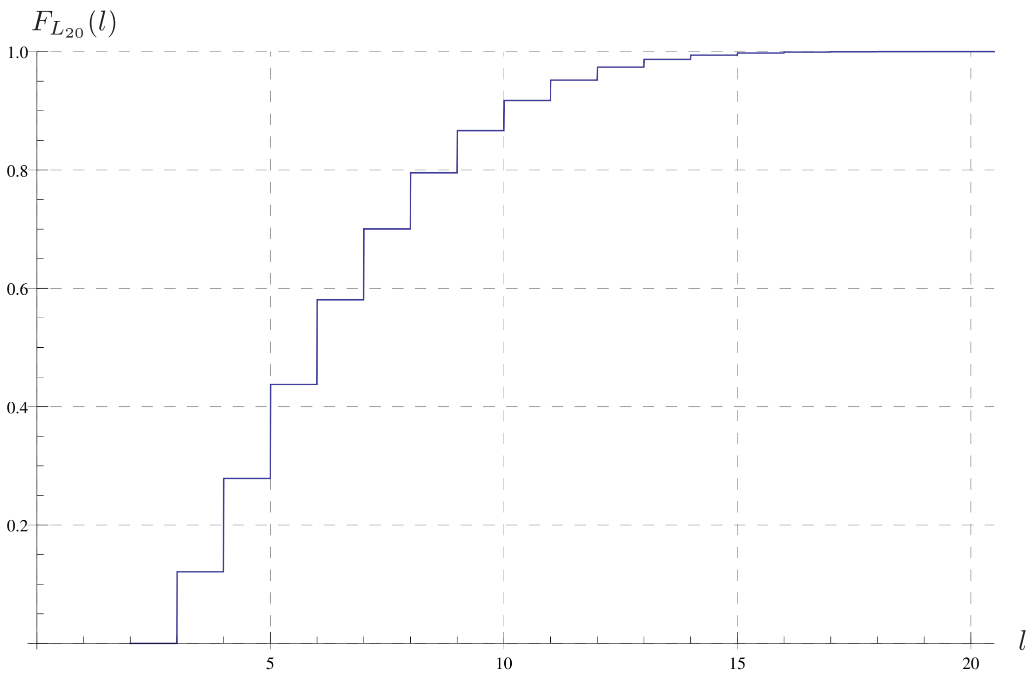


Figure 5.8: The cumulative distribution function of L_{20}.

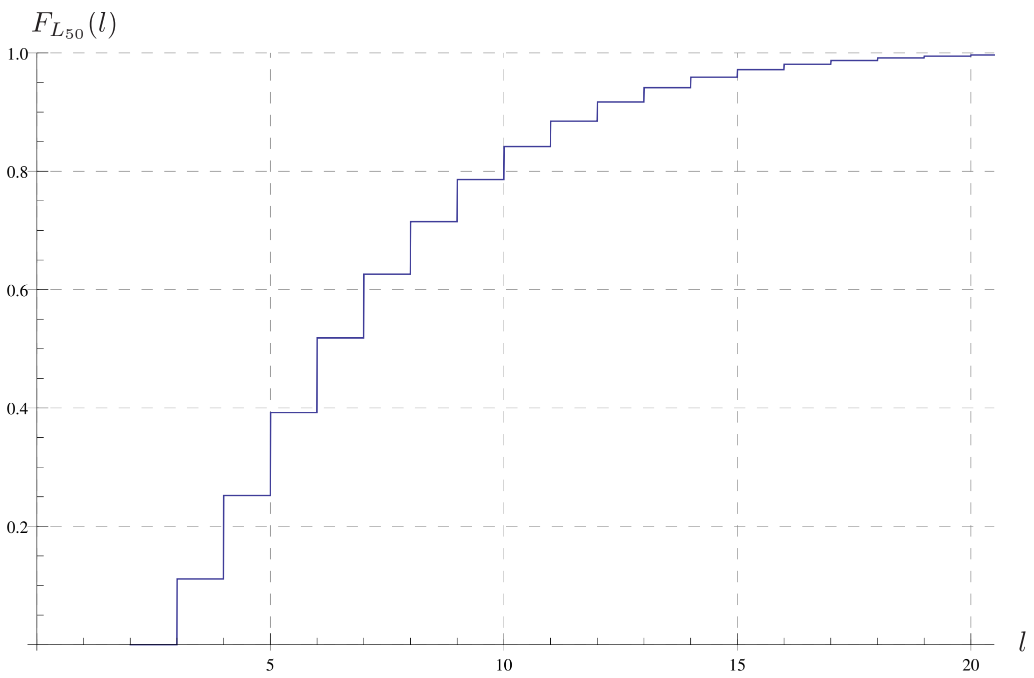


Figure 5.9: The cumulative distribution function of L_{50}.

5.2 Distribution of a random number of vertices

In this section we wish to obtain a similar result of convergence in distribution for random total number of vertices as we did in previous section for random boundary length. However, since the classes of triangulations with a fixed boundary length are countable infinite, we can not construct uniform measures on the classes. Nevertheless, as the choice of the uniform measure in the previous section, also the choice made here can be seen as the natural one. For more details the reader may consult the paper [3] by Angel and Schramm.
For defining probability measures on the classes at hand, we shall first prove the convergence of a series, which shall be used in our definition of the desired probability measures. In the proof we shall use Raabe's test, which is an extension of the ratio test and applicable when the ratio test fails to be conclusive.

Lemma 5.8 (Raabe's test)**.** *Let us consider a series*

$$\sum_{n=0}^{\infty} a_n.$$

If

$$\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) = R > 1,$$

then the series converges.

Proof. Let p be such that $R > p > 1$. By expanding $(1+\frac{1}{n})^p$ about infinity with respect to n, we obtain

$$\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) > p = \lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^p - 1\right).$$

Implying that there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{a_n}{a_{n+1}}\right| > \left(1+\frac{1}{n}\right)^p = \left(\frac{n+1}{n}\right)^p$$

when $n \geq n_0$. Hence

$$\begin{aligned}
\sum_{n=n_0}^{\infty} a_n &= a_{n_0} \sum_{n=n_0}^{\infty} \frac{a_{n_0+1}}{a_{n_0}} \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_n}{a_{n-1}} \\
&< a_{n_0} \sum_{n=n_0}^{\infty} \left(\frac{n_0}{n_0+1}\right)^p \left(\frac{n_0+1}{n_0+2}\right)^p \cdots \left(\frac{n-1}{n}\right)^p \\
&= a_{n_0} n_0^p \sum_{n=n_0}^{\infty} \left(\frac{1}{n}\right)^p < \infty,
\end{aligned}$$

since p was greater than one.

□

Lemma 5.9. *Let us set $\alpha = \frac{27}{256}$. Then*

$$Z_l(\alpha) = \sum_{n=0}^{\infty} \alpha^{n+l} D(n+l, l) < \infty \qquad \textit{for every } l \geq 3.$$

Proof. The value of the parameter α is in the sense critical that the test quantities of the ratio test and the root test, when applied to $Z_l(\alpha)$, equal to one and hence the both tests are inconclusive. Therefore we shall use

Raabe's test stated in the previous lemma. By using the definition of $D(V, l)$ (Equation 5.1) to the terms of $Z_l(\alpha)$ we obtain

$$\begin{aligned}\left|\frac{a_n}{a_{n+1}}\right| &= \frac{1}{\alpha} \cdot \frac{(n+1)\prod_{k=0}^{2}(3n+2l-k)}{\prod_{k=1}^{4}(4n+2l-k)} \\ &= \frac{\left(1+\frac{1}{n}\right)\prod_{k=0}^{2}\left(3+\frac{2l-k}{n}\right)}{\alpha\prod_{k=1}^{4}\left(4+\frac{2l-k}{n}\right)}.\end{aligned}$$

Hence

$$\begin{aligned}& n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \\ &= \frac{(n+1)\prod_{k=0}^{2}\left(3+\frac{2l-k}{n}\right) - \alpha(4n+2l-1)\prod_{k=2}^{4}\left(4+\frac{2l-k}{n}\right)}{\alpha\prod_{k=1}^{4}\left(4+\frac{2l-k}{n}\right)},\end{aligned} \tag{5.8}$$

where the limit of the denominator is

$$\alpha \cdot 4^4 = 27 \quad \text{as } n \to \infty. \tag{5.9}$$

The leading terms of the numerator in Equation 5.8 cancel each other out, whereas the constant term of the numerator is equal to

$$\begin{aligned}3^3 + 9\sum_{k=0}^{2}(2l-k) - 4^3 \cdot \alpha \sum_{k=1}^{4}(2l-k) &= 27 - 9 \cdot 3 + \frac{27}{4} \cdot 10 \\ &= \frac{135}{2}.\end{aligned} \tag{5.10}$$

Hence, by Equations 5.9 and 5.10, the limit of expression 5.8 is equal to $\frac{5}{2}$ and therefore by Raabe's test, the statement of the lemma holds. □

Let us fix l and define a class

$$\mathcal{T}^{(l)} = \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{T}_{n,l-3}$$

of triangulations having l boundary vertices. What we want to do next is to define a probability measure on the class. By the proof of the previous lemma we know that the radius of convergence of the series $Z_l(x)$ equals to α. Hence, if we would use any larger weighting than α, the series would diverge. From the root test follows that

$$\limsup_{V \to \infty} \sqrt[V]{D(V, l)} = \frac{1}{\alpha}.$$

Hence, if we would use any smaller weighting than α, the weight of the large triangulations would diminish. This motivates the following definition.

Definition 5.10 (Probability measure on $\mathcal{T}^{(l)}$)**.** Let us define a probability measure ν_l on $\mathcal{T}^{(l)}$ by

$$\nu_l\left(\{\mathscr{T}\}\right) = \frac{\alpha^{n+l}}{Z_l(\alpha)} \qquad \text{for every } \mathscr{T} \in \mathcal{T}_{n,l-3} \subset \mathcal{T}^{(l)}.$$

Then $\left(\mathcal{T}^{(l)}, \mathcal{P}\left(\mathcal{T}^{(l)}\right), \nu_l\right)$ is a probability space, where the sigma-algebra the events is the power set $\mathcal{P}\left(\mathcal{T}^{(l)}\right)$ of $\mathcal{T}^{(l)}$.

Next we shall define random variables on our probability spaces that are associated with the number of vertices of a triangulation. Since the class $\mathcal{T}^{(l)}$ consists of triangulations whose total number of vertices is at least equal to the boundary length l, if we wish to have any hope for convergence of the random variables as $l \to \infty$, we need to use an appropriate rescaling in our definition.

Definition 5.11 (Random total number of vertices on $\mathcal{T}^{(l)}$)**.** Let us define a random variable $X_l : \mathcal{T}^{(l)} \to \mathbb{R}^+$ returning a rescaled number of vertices of a triangulation by

$$X_l(\mathscr{T}) = \frac{V}{l^2} \qquad \text{for every } \mathscr{T} \in \mathcal{T}^{(l)}.$$

Then under the probability measure $P = \nu_l$ holds

$$P\left(X_l = \frac{V}{l^2}\right) = \frac{\alpha^V}{Z_l(\alpha)} D(V,l) \qquad \text{for every } V \in \mathbb{N}.$$

As in the previous section, we shall state the main theorem of the section already at this stage. The proof of the theorem can be found on page 84.

Theorem 5.12. *Let us consider the probability spaces $\left(\mathcal{T}^{(l)}, \mathcal{P}(\mathcal{T}^{(l)}), \nu_l\right)$ and the random variables $X_l : \mathcal{T}^{(l)} \longrightarrow \mathbb{R}^+$. There exists a non-degenerate random variable X such that X_l converges in distribution to X as $l \to \infty$.*

Again before proving the actual theorem, we shall put together all the necessary tools for the upcoming proof.

Definition 5.13. Let us define an extension for $D(V,l)$ to real numbers by

$$\tilde{D}(V,l) = \begin{cases} \frac{2\Pi(2l-3)\Pi(4V-2l-5)}{\Pi(l-3)\Pi(l-1)\Pi(V-l)\Pi(3V-l-3)}, & \text{when } V \geq l \geq 3. \\ 0, & \text{otherwise.} \end{cases}$$

Above Π is the pi function extending the factorial function analytically to the complex numbers except the negative integers. Therefore $\tilde{D}$ is continuous when $V > l > 3$. From the definition follows also that

$$\tilde{D}(V, l) = D(V, l)$$

for every $V, l \in \mathbb{N}$ with $V \geq l \geq 3$. The pi function can also be expressed in terms of the better-know gamma function Γ by $\Pi(z) = \Gamma(z + 1)$ for every complex number z except the negative integers. About the gamma function the reader may consult for example Andrews's book [2].

Lemma 5.14. *Let us assume that $x > 0$. Then*

$$\begin{aligned} \lim_{l\to\infty} f_l(x) &:= \lim_{l\to\infty} \tilde{D}(xl^2, l)\frac{\alpha^{xl^2} l^{\frac{9}{2}}}{\left(\frac{3}{4}\right)^l} \\ &= \frac{9\sqrt{\frac{3}{2}}}{2048\pi x^{\frac{5}{2}} e^{\frac{1}{6x}}} =: p(x), \end{aligned}$$

where $\alpha = \frac{27}{256}$.

Proof. Although we are now considering the pi function instead of the factorial function, Stirling's approximation (Proposition 5.1) still holds yielding

$$\lim_{l\to\infty} f_l(x) = \lim_{l\to\infty} \frac{4^l \alpha^{xl^2} l^{\frac{9}{2}} e}{3^l \pi} \cdot \frac{(2l-3)^{2l-\frac{5}{2}} (4xl^2 - 2l - 5)^{4xl^2 - 2l - \frac{9}{2}}}{(l-3)^{l-\frac{5}{2}} (l-1)^{l-\frac{1}{2}} (xl^2 - l)^{xl^2 - l + \frac{1}{2}} (3xl^2 - l - 3)^{3xl^2 - l - \frac{5}{2}}}, \tag{5.11}$$

where

$$\begin{cases} (2l-3)^{2l-\frac{5}{2}} &= (2l)^{2l-\frac{5}{2}} \left(1 - \frac{3}{2l}\right)^{2l-\frac{5}{2}} \\ (l-3)^{l-\frac{5}{2}} &= l^{l-\frac{5}{2}} \left(1 - \frac{3}{l}\right)^{l-\frac{5}{2}} \\ (l-1)^{l-\frac{1}{2}} &= l^{l-\frac{1}{2}} \left(1 - \frac{1}{l}\right)^{l-\frac{1}{2}} \\ (xl^2 - l)^{xl^2 - l + \frac{1}{2}} &= (xl^2)^{xl^2 - l + \frac{1}{2}} \left(1 - \frac{1}{xl}\right)^{-l+\frac{1}{2}} \left(1 - \frac{1}{xl}\right)^{xl^2} \end{cases} \tag{5.12}$$

and

$$\begin{aligned} \left(3xl^2 - l - 3\right)^{3xl^2 - l - \frac{5}{2}} &= \left(3xl^2 - l\right)^{3xl^2 - l - \frac{5}{2}} \left(1 - \frac{3}{3xl^2 - l}\right)^{3xl^2 - l - \frac{5}{2}} \\ &= \left(3xl^2\right)^{3xl^2 - l - \frac{5}{2}} \left(1 - \frac{1}{3xl}\right)^{-l-\frac{5}{2}} \\ &\quad \left(1 - \frac{1}{3xl}\right)^{3xl^2} \left(1 - \frac{3}{3xl^2 - l}\right)^{3xl^2 - l - \frac{5}{2}}. \end{aligned} \tag{5.13}$$

Similarly

$$\left(4xl^2-2l-5\right)^{4xl^2-2l-\frac{9}{2}} = \left(4xl^2\right)^{4xl^2-2l-\frac{9}{2}} \left(1-\frac{1}{2xl}\right)^{-2l-\frac{9}{2}} \left(1-\frac{1}{2xl}\right)^{4xl^2} \left(1-\frac{5}{4xl^2-2l}\right)^{4xl^2-2l-\frac{9}{2}}. \tag{5.14}$$

Next we shall substitute the latter expressions of Equations 5.12, 5.13 and 5.14 in Equation 5.11. By combining the first terms of the substituted expressions with the first fraction of expression 5.11, the exponents of l cancel each other out leaving

$$\frac{2^{2l}3^{3xl^2}e}{3^l 2^{8xl^2}\pi} \cdot \frac{2^{8xl^2-2l-\frac{23}{2}}}{3^{3xl^2-l-\frac{5}{2}}} x^{-\frac{5}{2}} = \frac{3^{\frac{5}{2}}e}{2^{\frac{23}{2}}\pi x^{\frac{5}{2}}}. \tag{5.15}$$

In addition, we shall combine the terms of the substituted expressions that converge to an exponential term. These terms are the ones with the order of l in the exponent of the term matching to the order of l in the fraction of the term. As the terms are combined, we obtain that its limit equals to

$$\frac{e^{-3}e^{-5}e^{-\frac{1}{x}}}{e^{-3}e^{-1}e^{-\frac{1}{x}}e^{-3}e^{\frac{1}{3x}}} = e^{-1-\frac{1}{3x}}. \tag{5.16}$$

We still have to consider the remaining terms of the substituted expressions that reads

$$\frac{\left(1-\frac{1}{2xl}\right)^{4xl^2}}{\left(1-\frac{1}{xl}\right)^{xl^2}\left(1-\frac{1}{3xl}\right)^{3xl^2}} = \left[\frac{\left(1-\frac{1}{2xl}\right)^4}{\left(1-\frac{1}{xl}\right)\left(1-\frac{1}{3xl}\right)^3}\right]^{xl^2}. \tag{5.17}$$

By expanding with respect to $l=\infty$ we obtain

$$\begin{aligned} \left(1-\tfrac{1}{2xl}\right)^4 &= 1-\tfrac{2}{xl}+\tfrac{3}{2x^2l^2}+\mathcal{O}\left(\tfrac{1}{(xl)^3}\right) \\ \left(1-\tfrac{1}{3xl}\right)^3 &= 1-\tfrac{1}{xl}+\tfrac{1}{3x^2l^2}+\mathcal{O}\left(\tfrac{1}{(xl)^3}\right). \end{aligned}$$

Hence

$$\begin{aligned}\left[\frac{\left(1-\frac{1}{2xl}\right)^4}{\left(1-\frac{1}{xl}\right)\left(1-\frac{1}{3xl}\right)^3}\right]^{xl^2} &= \left[\frac{1-\frac{2}{xl}+\frac{3}{2x^2l^2}+\mathcal{O}\left(\frac{1}{(xl)^3}\right)}{1-\frac{2}{xl}+\frac{4}{3x^2l^2}+\mathcal{O}\left(\frac{1}{(xl)^3}\right)}\right]^{xl^2} \\ &= \left(1+\frac{1}{6x^2l^2}+\mathcal{O}\left(\frac{1}{(xl)^3}\right)\right)^{xl^2} \\ &= \left(1+\frac{\frac{1}{6x}+\mathcal{O}\left(\frac{1}{x^2l}\right)}{xl^2}\right)^{xl^2} \to e^{\frac{1}{6x}}\end{aligned} \tag{5.18}$$

for every $x > 0$ as $l \to \infty$. By combining the results 5.15, 5.16 and 5.18, we may conclude that

$$\begin{aligned}\lim_{l\to\infty} f_l(x) &= \frac{9\sqrt{\frac{3}{2}}}{2048\pi x^{\frac{5}{2}} e^{\frac{1}{6x}}} \\ &= p(x).\end{aligned} \tag{5.19}$$

□

Lemma 5.15. *The convergence of Lemma 5.14 is dominated in $L^1(1,\infty)$.*

Proof. Let us assume that $x > 1$. By using Stirling's approximation (Proposition 5.1) with error terms, we may write

$$f_l(x) = \frac{4^l \alpha^{xl^2} l^{\frac{9}{2}} e}{3^l \pi} q_l(x)\delta_l(x), \tag{5.20}$$

where the function $q_l(x)$ denotes the latter rational of Equation 5.11 and $\delta_l(x)$ is the error function resulting from Stirling's approximation. The error function reads

$$\delta_l(x) = \frac{\left(1+\mathcal{O}\left(\frac{1}{2l-3}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{4xl^2-2l-5}\right)\right)}{\left(1+\mathcal{O}\left(\frac{1}{l-3}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{l-1}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{xl^2-l}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{3xl^2-l-3}\right)\right)}.$$

Since $x > 1$, we may fix $x = 1$ in the error terms above obtaining

$$\begin{aligned}\delta_l(x) &= \frac{1+\mathcal{O}\left(\frac{1}{l}\right)}{1+\mathcal{O}\left(\frac{1}{l}\right)} \\ &= 1+\mathcal{O}\left(\frac{1}{l}\right).\end{aligned} \tag{5.21}$$

In the proof of the Lemma 5.14 we made decompositions of the terms of $q_l(x)$ (see Equations 5.12, 5.13 and 5.14). Then we combined the first terms of the decompositions with the rational term on the left side of the function $q_l(x)$ in Equality 5.20. After making these combinations, let us denote the remaining part of $q_l(x)$ with $q_l'(x)$. By Equations 5.15 and 5.21, we may now deduce from Equation 5.20 that

$$f_l(x) < \frac{C}{x^{\frac{5}{2}}} q_l'(x)$$

when l is large enough. Above C is a constant consisting of the constant part of the expression 5.15 multiplied with some larger than one constant coming from the estimate 5.21. Let us next consider the part of $g_l'(x)$ whose limit is independent of x. Again by earlier calculations (see Equation 5.16), we know that the limit equals to e^{-1}. We shall redefine the constant C in such a way that it takes care of our considerations about this part of $g_l'(x)$. Now we may write (see Equations 5.12, 5.13 and 5.14)

$$f_l(x) < \frac{C}{x^{\frac{5}{2}}} \cdot \frac{\left(1-\frac{1}{2xl}\right)^{-2l-\frac{9}{2}} \left(1-\frac{1}{2xl}\right)^{4xl^2}}{\left(1-\frac{1}{xl}\right)^{-l+\frac{1}{2}} \left(1-\frac{1}{xl}\right)^{xl^2} \left(1-\frac{1}{3xl}\right)^{-l-\frac{5}{2}} \left(1-\frac{1}{3xl}\right)^{3xl^2}} \tag{5.22}$$

when l is large enough. Let us consider the terms above of form

$$\left(1-\frac{a}{xk}\right)^{-k-b} = \exp\left((-k-b)\log\left(1-\frac{a}{xk}\right)\right),$$

where a is a positive real number and b is a real number, and $k \to \infty$ as $l \to \infty$. Since $x > 1$, we may expand the logarithmic function about one obtaining

$$\begin{aligned}
\exp\left((-k-b)\log\left(1-\frac{a}{xk}\right)\right) &= \exp\left((-k-b)\left(-\frac{a}{xk}+\mathcal{O}\left(\frac{a^2}{x^2k^2}\right)\right)\right) \\
&= \exp\left(\frac{a(k+b)}{xk}\right)\exp\left(\mathcal{O}\left(\frac{a^2}{x^2k}\right)\right) \\
&\leq \exp\left(a+\frac{b}{k}\right)\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right) \\
&\leq C_{a,b}\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right)
\end{aligned}$$

when k is large enough. The first equality follows from the expansion of the logarithmic function. The first inequality follows from the expansion of the exponential function and from the fact that $x > 1$. The second inequality holds for some positive constant $C_{a,b}$ depending on a and b. When the estimate above is applied to the terms of Inequality 5.22 we obtain after redefining the constant C that

$$f_l(x) < \frac{C}{x^{\frac{5}{2}}} \cdot \frac{\left(1 - \frac{1}{2xl}\right)^{4xl^2}}{\left(1 - \frac{1}{xl}\right)^{xl^2} \left(1 - \frac{1}{3xl}\right)^{3xl^2}} \tag{5.23}$$

when l is large enough. By Equation 5.18 we may write

$$\begin{aligned}
\frac{\left(1 - \frac{1}{2xl}\right)^{4xl^2}}{\left(1 - \frac{1}{xl}\right)^{xl^2} \left(1 - \frac{1}{3xl}\right)^{3xl^2}} &= \exp\left(xl^2 \log\left(1 + \frac{\frac{1}{6x} + \mathcal{O}\left(\frac{1}{x^2 l}\right)}{xl^2}\right)\right) \\
&= \exp\left(\frac{1}{6x} + \mathcal{O}\left(\frac{1}{x^2 l}\right)\right) \\
&\leq \exp\left(\frac{1}{6}\right) \exp\left(\mathcal{O}\left(\frac{1}{l}\right)\right) \\
&\leq \tilde{C} e^{\frac{1}{6}}
\end{aligned}$$

for some positive constant $\tilde{C}$ when l is large enough. The second equality above follows from the expansion of the logarithmic function. Now after redefining the constant in Inequality 5.23 we obtain

$$f_l(x) < \frac{C}{x^{\frac{5}{2}}} \tag{5.24}$$

when l is large enough and thus we have found an integrable upper bound for $f_l(x)$ when $x \in (1, \infty)$. $\square$

Lemma 5.16. *For every $c > 0$ there exists strictly positive constants C_+ and C_- such that*

$$C_- \exp\left[\left(n + \frac{1}{2}\right) \log(n + c) - n\right] \leq n! \leq C_+ \exp\left[\left(n + \frac{1}{2}\right) \log(n + c) - n\right]$$

for every non-negative integer n.

Proof. Let $c > 0$. Since

$$\frac{\sqrt{n + c}}{\sqrt{n}} \to 1$$

and

$$\begin{aligned}\frac{(n+c)^n}{n^n} &= e^{n\log\left(1+\frac{c}{n}\right)}\\ &= e^{n\left(\frac{c}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)} \to e^c,\end{aligned}$$

we obtain by Stirling's approximation (Proposition 5.1) that

$$\frac{n!}{(n+c)^{n+\frac{1}{2}}\, e^{-n}} \to \sqrt{2\pi}e^{-c}.$$

Hence, for every $\epsilon > 0$ there exists n_ϵ such that

$$(\sqrt{2\pi}e^{-c}-\epsilon)(n+c)^{n+\frac{1}{2}}e^{-n} \le n! \le (\sqrt{2\pi}e^{-c}+\epsilon)(n+c)^{n+\frac{1}{2}}e^{-n}$$

when $n \ge n_\epsilon$. Let us define constants C_+ and C_- in such a way that they cover also the finite number of cases, where $n < n_\epsilon$. After that, the claim is obtained by writing the inequalities in terms of the exponential function. □

Lemma 5.17. *The function defined by*

$$f(V,l) = D(V,l)\frac{\alpha^V l^{\frac{9}{2}}}{\left(\frac{3}{4}\right)^l} \tag{5.25}$$

is bounded.

Proof. By applying the lower bound of Lemma 5.16 to the denominator and the upper bound to the numerator of function $D(V,l)$, and by writing the remaining terms in terms of the exponential function we obtain

$$\begin{aligned}f(V,l) \le \tilde{C}\exp\Bigg(1 &+ \left(2l-\frac{5}{2}\right)\log\left(2l-3+c\right)\\ &+ \left(4V-2l-\frac{9}{2}\right)\log\left(4V-2l-5+c\right) - \left(l-\frac{5}{2}\right)\log\left(l-3+c\right)\\ &- \left(l-\frac{1}{2}\right)\log\left(l-1+c\right) - \left(V-l+\frac{1}{2}\right)\log\left(V-l+c\right)\\ &- \left(3V-l-\frac{5}{2}\right)\log\left(3V-l-3+c\right)\Bigg)\cdot\\ &\exp\left(3V\log 3 - 8V\log 2 + \frac{9}{2}\log l + 2l\log 2 - l\log 3\right),\end{aligned} \tag{5.26}$$

where $\tilde{C}$ is some positive constant. Our plan is to find an upper bound for the expression above that holds for every $V \geq V_0 \in \mathbb{N}$ and $l \in \{3, 4, ..., V\}$. Since for every $V \leq V_0$ there exists only a finite number of values of l yielding a non-zero value when mapped by D, this would be sufficient to prove that $f(V, l)$ is bounded.

We shall first take the exponent of the expression above and omit the constant one. After that we shall make the following substitutions

$$\begin{cases} \log(2l - 3 + c) & = \log\left(1 + \frac{c-3}{2l}\right) + \log 2 + \log l \\ \log(4V - 2l - 5 + c) & = \log\left(1 - \frac{l}{2V} + \frac{c-5}{4V}\right) + 2\log 2 + \log V \\ \log(l - 3 + c) & = \log\left(1 + \frac{c-3}{l}\right) + \log l \\ \log(l - 1 + c) & = \log\left(1 + \frac{c-1}{l}\right) + \log l \\ \log(V - l + c) & = \log\left(1 + \frac{c-l}{V}\right) + \log V \\ \log(3V - l - 3 + c) & = \log\left(1 - \frac{l}{3V} + \frac{c-3}{3V}\right) + \log 3 + \log V. \end{cases} \tag{5.27}$$

What we wish to bound can now be written as

$$l g_l(V, l, c) + V g_V(V, l, c) + g(V, l, c),$$

where

$$\begin{aligned} g_l(V, l, c) = & \, 2\log\left(1 + \frac{c-3}{2l}\right) - 2\log\left(1 - \frac{l}{2V} + \frac{c-5}{4V}\right) \\ & - \log\left(1 + \frac{c-3}{l}\right) - \log\left(1 + \frac{c-1}{l}\right) \\ & + \log\left(1 + \frac{c-l}{V}\right) + \log\left(1 - \frac{l}{3V} + \frac{c-3}{3V}\right), \end{aligned}$$

$$\begin{aligned} g_V(V, l, c) = & \, 4\log\left(1 - \frac{l}{2V} + \frac{c-5}{4V}\right) - \log\left(1 + \frac{c-l}{V}\right) \\ & - 3\log\left(1 - \frac{l}{3V} + \frac{c-3}{3V}\right) \end{aligned}$$

and

$$\begin{aligned}
g(V,l,c) = {} & \frac{9}{2}\log l - \frac{5}{2}\left(\log\left(1+\frac{c-3}{2l}\right) + \log 2 + \log l\right) \\
& - \frac{9}{2}\left(\log\left(1-\frac{l}{2V}+\frac{c-5}{4V}\right) + 2\log 2 + \log V\right) \\
& + \frac{5}{2}\left(\log\left(1+\frac{c-3}{l}\right) + \log l\right) + \frac{1}{2}\left(\log\left(1+\frac{c-1}{l}\right) + \log l\right) \\
& - \frac{1}{2}\left(\log\left(1+\frac{c-l}{V}\right) + \log V\right) \\
& + \frac{5}{2}\left(\log\left(1-\frac{l}{3V}+\frac{c-3}{3V}\right) + \log 3 + \log V\right).
\end{aligned}$$

When multiplied with l, the first, third and fourth term of $g_l(V,l,c)$ converge to some constant by the limit definition of the exponential function. Therefore they are also bounded. In addition to the constant terms, we may also omit every other logarithmic term of function $g(V,l,c)$ with the exception of the penultimate one. This can be done since when $l \geq 4$, $V \geq l$ and c is a strictly positive constant, all those terms are bounded. After these observations, what we wish to bound can now be written as

$$\begin{aligned}
B(V,l,c) := {} & 5\log l - \frac{5}{2}\log V - \frac{1}{2}\log\left(1-\frac{l}{V}+\frac{c}{V}\right) \\
& + l\left(\log\left(1-\frac{l}{3V}+\frac{c-3}{3V}\right) + \log\left(1+\frac{c-l}{V}\right)\right. \\
& \left. -2\log\left(1-\frac{l}{2V}+\frac{c-5}{4V}\right)\right) \\
& + V\left(4\log\left(1-\frac{l}{2V}+\frac{c-5}{4V}\right) - \log\left(1+\frac{c-l}{V}\right)\right. \\
& \left. -3\log\left(1-\frac{l}{3V}+\frac{c-3}{3V}\right)\right).
\end{aligned}$$

At this point we shall fix $c = 1$ to make our upcoming calculations a bit less cumbersome. We shall also define an auxiliary function that we shall utilize as we finish the proof of the lemma as follows

$$h(V,\theta) = B(V,\theta V,1), \tag{5.28}$$

where $\frac{3}{V} \leq \theta \leq 1$. Next we shall show that the auxiliary function has an unique maximum with respect to θ on the given interval. The first and the second derivative of the auxiliary function can be written after some simplifications as

$$\begin{aligned}\frac{\partial h}{\partial \theta}(V,\theta) = \frac{5}{\theta} &+ V\left(-\frac{2}{2+V(\theta-3)} + \frac{4}{2+V(\theta-2)} + \frac{1}{-2+2V(\theta-1)}\right)\\ &+ V\left(\log\left(1-\frac{\theta}{3}-\frac{2}{3V}\right) + \log\left(1-\theta+\frac{1}{V}\right)\right.\\ &\left.-2\log\left(1-\frac{\theta}{2}-\frac{1}{V}\right)\right),\end{aligned} \tag{5.29}$$

$$\begin{aligned}\frac{\partial^2 h}{\partial \theta^2}(V,\theta) = -\frac{5}{\theta^2} + V^2 &\left(\frac{2}{(2+V(\theta-3))^2}\right.\\ &-\frac{4}{(2+V(\theta-2))^2} - \frac{1}{2(-1+V(\theta-1))^2}\\ &\left.+\frac{1}{2+V(\theta-3)} - \frac{1}{1+V(1-\theta)} - \frac{2}{2+V(\theta-2)}\right).\end{aligned}$$

By omitting terms that are always negative from the expression of the second derivative we obtain

$$\begin{aligned}\frac{\partial^2 h}{\partial \theta^2}(V,\theta) < V^2 &\left(\frac{2}{(2+V(\theta-3))^2} + \frac{1}{2+V(\theta-3)}\right.\\ &\left.+\frac{1}{-1+V(\theta-1)} - \frac{2}{2+V(\theta-2)}\right).\end{aligned}$$

Let us first consider the case where $\frac{3}{V} \leq \theta \leq 1-\epsilon$ with some fixed $\epsilon > 0$. Now the first rational term is of order $\frac{1}{V^2}$, whereas the others are of order $\frac{1}{V}$ and hence dominant. By expanding the dominant terms about $V = \infty$ we obtain

$$\begin{aligned}&\frac{1}{2+V(\theta-3)} + \frac{1}{-1+V(\theta-1)} - \frac{2}{2+V(\theta-2)}\\ =&\frac{1}{V}\left(\frac{1}{\theta-3}+\frac{1}{\theta-1}-\frac{2}{\theta-2}\right) + \mathcal{O}\left(\frac{1}{V^2}\right),\end{aligned}$$

where the error term is uniform in θ, since $\theta \leq 1-\epsilon$. A simple expansion gives us

$$\frac{1}{\theta-3}+\frac{1}{\theta-1}-\frac{2}{\theta-2} = \frac{2}{(\theta-3)(\theta-2)(\theta-1)} < 0$$

implying that the second derivative is negative when $\theta \in \left[\frac{3}{V}, 1-\epsilon\right]$ and V is large enough.
Let us now assume that $1-\epsilon < \theta \leq 1$ and examine the first derivative presented in Equation 5.29. First we notice that

$$\begin{aligned} &V\left(-\frac{2}{2+V(\theta-3)}+\frac{4}{2+V(\theta-2)}+\frac{1}{-2+2V(\theta-1)}\right) \\ &= -\frac{2}{\frac{2}{V}+\theta-3}+\frac{4}{\frac{2}{V}+\theta-2}+\frac{1}{-\frac{2}{V}+2(\theta-1)}, \end{aligned} \tag{5.30}$$

where all but the last term are bounded. However, the unbounded term is always negative. In addition, when we take a look at the logarithmic terms of Equation 5.29 we notice that the first one is always negative. Therefore when $\theta \in (1-\epsilon, 1]$, the following bound holds

$$\frac{\partial h}{\partial \theta}(V,\theta) < M + V\left(\log\left(1-\theta+\frac{1}{V}\right) - 2\log\left(1-\frac{\theta}{2}-\frac{1}{V}\right)\right)$$

with some positive constant M. The latter logarithmic term of the expression above is bounded and for the first one holds

$$\log\left(1-\theta+\frac{1}{V}\right) < \log\left(\epsilon+\frac{1}{V}\right).$$

Thus by selecting ϵ small enough, the first derivative is negative when $\theta \in (1-\epsilon, 1]$ and V is large enough.

Let us next fix $\theta = \frac{1}{\sqrt{V}}$ in Equation 5.29 obtaining

$$\begin{aligned} \frac{\partial h}{\partial \theta}(V, \frac{1}{\sqrt{V}}) = 5\sqrt{V} + V &\left(-\frac{2}{2+V(\frac{1}{\sqrt{V}}-3)}+\frac{4}{2+V(\frac{1}{\sqrt{V}}-2)}\right. \\ &\left.+\frac{1}{-2+2V(\frac{1}{\sqrt{V}}-1)}\right) + V\left(\log\left(1-\frac{1}{3\sqrt{V}}-\frac{2}{3V}\right)\right. \\ &\left.+\log\left(1-\frac{1}{\sqrt{V}}+\frac{1}{V}\right)-2\log\left(1-\frac{1}{2\sqrt{V}}-\frac{1}{V}\right)\right). \end{aligned} \tag{5.31}$$

By Equation 5.30 it is easy to see that

$$V\left(-\frac{2}{2+V(\frac{1}{\sqrt{V}}-3)}+\frac{4}{2+V(\frac{1}{\sqrt{V}}-2)}+\frac{1}{-2+2V(\frac{1}{\sqrt{V}}-1)}\right) \to -\frac{11}{6}, \tag{5.32}$$

as $V \to \infty$. Next we shall expand the logarithmic terms of Equation 5.31 about infinity obtaining

$$\begin{aligned}
&V\left(\log\left(1-\frac{1}{3\sqrt{V}}-\frac{2}{3V}\right)+\log\left(1-\frac{1}{\sqrt{V}}+\frac{1}{V}\right)\right.\\
&\qquad\left.-2\log\left(1-\frac{1}{2\sqrt{V}}-\frac{1}{V}\right)\right)\\
&=V\left(-\frac{1}{3\sqrt{V}}-\frac{1}{\sqrt{V}}+\frac{1}{\sqrt{V}}+\mathcal{O}\left(\frac{1}{V}\right)\right)\\
&=-\frac{1}{3}\sqrt{V}+\mathcal{O}(1).
\end{aligned} \tag{5.33}$$

By Equations 5.32 and 5.33 the first derivative at $\theta = \frac{1}{\sqrt{V}}$ (Equation 5.31) is positive when V is large enough. Identical examination as above will show that the first derivative at $\theta = \frac{10}{\sqrt{V}}$ is negative when V is large enough. Combined with negativity of the second derivative when $\theta \in \left[\frac{3}{V}, 1-\epsilon\right]$ and negativity of the first derivative when $\theta \in (1-\epsilon, 1]$ we have proven that

$$\frac{1}{\sqrt{V}} < \theta_{\max}(V) := \underset{\theta\in\left[\frac{3}{V},1\right]}{\arg\max}\, h(V,\theta) < \frac{10}{\sqrt{V}} \tag{5.34}$$

when V is large enough.

Now we have all the necessary tools to finish the proof of the lemma. Let us argue by contradiction and assume that $B(V,l,1)$ is not bounded. Then there exists a sequence $((V_n,l_n))_{n\in\mathbb{N}}$, where $V_n \to \infty$ and $l_n \in \{3,...,V_n\}$, such that

$$B(V_n,l_n,1) = h\left(V_n,\frac{l_n}{V_n}\right) \to \infty.$$

Then also

$$h(V_n,\theta_{\max}(V_n)) \to \infty$$

implying that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $1 < a_n < 10$ and

$$h\left(V_n,\frac{a_n}{\sqrt{V_n}}\right) \to \infty.$$

Since the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, there exists a converging subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that

$$a_{n_k} \to a \in [1,10] \qquad \text{and} \qquad h\left(V_{n_k},\frac{a_{n_k}}{\sqrt{V_{n_k}}}\right) \to \infty.$$

Let us now take a careful look at the steps of the proof at the same time acknowledging particularly how the bound $B(V, l, 1)$ was obtained. We started our proof by defining the function $f(V, l)$ in Equation 5.25. When $V = \sigma l^2$ is fixed, by Lemma 5.14 the limit of the extended function f_l is known as $l \to \infty$. By the proof of Lemma 5.16, we know that the limit behaviour of the bounds used in Inequality 5.26 is essentially (up to a constant multiplier) the same as the limit behaviour of Stirling's approximation (Proposition 5.1) used in Equation 5.11. The function $B(V, l, 1)$ was obtained from $f(V, l)$ by taking the exponent of the bound of Inequality 5.26 and after that, by omitting the constants and the terms that converge to a constant as $V = \sigma l^2$ is fixed. By putting this all together we may deduce that

$$\begin{aligned}\lim_{V\to\infty} h\left(V, \frac{1}{\sqrt{\sigma V}}\right) &= \lim_{V\to\infty} B\left(V, \sqrt{\frac{V}{\sigma}}, 1\right) \\ &= \lim_{l\to\infty} B(\sigma l^2, l, 1) \\ &= \hat{C} - \frac{1}{6\sigma} - \log \sigma^{\frac{5}{2}},\end{aligned}$$

where $\hat{C}$ is some constant. Hence

$$h\left(V_{n_k}, \frac{a_{n_k}}{\sqrt{V_{n_k}}}\right) \to \hat{C} - \frac{a^2}{6} + \log a^5$$

contradicting the assumption that $B(V, l, 1)$ is not bounded. □

Corollary 5.18.

$$\begin{aligned}\lim_{l\to\infty} \sum_{\substack{V\in\mathbb{N}\\ V\le\sigma l^2}} f_l\left(\frac{V}{l^2}\right)\cdot\frac{1}{l^2} &= \lim_{l\to\infty} \sum_{\substack{V\in\mathbb{N}\\ V\le\sigma l^2}} \frac{\alpha^V D(V, l)}{\lambda(l)}\cdot\frac{1}{l^2} \\ &= \int_0^\sigma p(x)\; dx,\end{aligned}$$

where

$$\lambda(l) = \frac{\left(\frac{3}{4}\right)^l}{l^{\frac{9}{2}}}.$$

Proof. The first equality follows straight from the definition of f_l. Let us consider a piecewise constant function

$$\tilde{f}_l(x) = \begin{cases} f_l\left(\frac{1}{l^2}\right), & x \in \left(0, \frac{1}{l^2}\right] \\ f_l\left(\frac{2}{l^2}\right), & x \in \left(\frac{1}{l^2}, \frac{2}{l^2}\right] \\ \vdots & \\ f_l(\frac{\lfloor \sigma l^2 \rfloor}{l^2}), & x \in \left(\frac{\lfloor \sigma l^2 \rfloor - 1}{l^2}, \frac{\lfloor \sigma l^2 \rfloor}{l^2}\right] \\ f_l(\frac{\lfloor \sigma l^2 \rfloor + 1}{l^2}), & x \in \left(\frac{\lfloor \sigma l^2 \rfloor}{l^2}, \frac{\lfloor \sigma l^2 \rfloor + 1}{l^2}\right] \\ \vdots & \end{cases}$$

where $\lfloor * \rfloor$ denotes the floor function. We shall prove that $\tilde{f}_l(x)$ has the very same limit function as $f_l(x)$ has, namely $p(x)$.
Let us choose an arbitrary point $x_0 \in (0, \infty)$. Let us first assume that l is such large that $x_0 > \frac{1}{l}$ letting us use the continuity of $f_l(x)$ at x_0. That is, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f_l(x) - f_l(x_0)| < \frac{\epsilon}{2} \quad \text{when } |x - x_0| < \delta.$$

Let us assume next that l is such large that

$$\frac{1}{l^2} < \delta \quad \text{and} \quad |f_l(x_0) - p(x_0)| < \frac{\epsilon}{2}.$$

Then by the triangle inequality and the definition of $\tilde{f}_l(x)$

$$\begin{aligned} \left|\tilde{f}_l(x_0) - p(x_0)\right| &\le \left|\tilde{f}_l(x_0) - f_l(x_0)\right| + |f_l(x_0) - p(x_0)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence

$$\lim_{l \to \infty} \tilde{f}_l(x) = p(x) \qquad \text{for every } x \in (0, \infty).$$

The definite integral of $\tilde{f}_l(x)$ from 0 to σ reads

$$\int_0^\sigma \tilde{f}_l(x)\, dx = f_l\left(\frac{\lfloor \sigma l^2 \rfloor + 1}{l^2}\right)\left(\sigma - \frac{\lfloor \sigma l^2 \rfloor}{l^2}\right) + \sum_{\substack{V \in \mathbb{N} \\ V \le \sigma l^2}} f_l\left(\frac{V}{l^2}\right) \cdot \frac{1}{l^2}.$$

By the definitions of $\tilde{f}_l$, f_l and $\tilde{D}$, the image of $\tilde{f}_l(x)$ is a subset of the image of the bounded function $f(V, l)$. Hence the sequence $\left(\tilde{f}_l(x)\right)_l$ is uniformly bounded. In addition, since the bound of Inequality 5.24 is strictly decreasing, it holds for $\tilde{f}_l(x)$ when $x > 1$ and l is large enough. By combining these two facts, we obtain an upper bound in $L^1(0, \infty)$ for $\tilde{f}_l(x)$ and hence by Lebesgue's dominated convergence theorem and the equation above we may conclude that

$$\begin{aligned}
\int_0^\sigma p(x)\, dx &= \lim_{l\to\infty} \int_0^\sigma \tilde{f}_l(x)\, dx \\
&= p(\sigma) \lim_{l\to\infty} \left(\sigma - \frac{\lfloor \sigma l^2 \rfloor}{l^2} \right) + \lim_{l\to\infty} \sum_{\substack{V\in\mathbb{N} \\ V\le\sigma l^2}} f_l\left(\frac{V}{l^2}\right) \cdot \frac{1}{l^2} \\
&= \lim_{l\to\infty} \sum_{\substack{V\in\mathbb{N} \\ V\le\sigma l^2}} f_l\left(\frac{V}{l^2}\right) \cdot \frac{1}{l^2}.
\end{aligned}$$

□

Now we are able to prove Theorem 5.12 by means of pointwise convergence of the cumulative distribution functions.

Proof of Theorem 5.12. Let us first define the cumulative distribution functions F_l for the random variables X_l.

$$\begin{aligned}
F_l(\sigma) &= P\left(X_l = \frac{V_l}{l^2} \le \sigma \right) \\
&= P\left(V_l \le \sigma l^2 \right) \\
&= \sum_{\substack{n\in\mathbb{N}\cup\{0\} \\ n\le\sigma l^2 - l}} P\left(V_l = n + l \right) \\
&= \sum_{\substack{n\in\mathbb{N}\cup\{0\} \\ n\le\sigma l^2 - l}} \frac{\alpha^{n+l}}{Z_l(\alpha)} D(n+l, l).
\end{aligned} \tag{5.35}$$

We notice that $F_l(\sigma) = 0$ for every $\sigma < \frac{1}{l}$ and by the definition of $Z_l(\alpha)$ (see Lemma 5.9) $F_l(\infty) = 1$ for every $l \ge 3$, as it should be. By changing the index of summation of Corollary 5.18 from V to $n = V - l$ we obtain

$$\begin{aligned}\frac{1}{l^2\lambda(l)}\sum_{\substack{V\in\mathbb{N}\\ V\le\sigma l^2}}\alpha^V D(V,l) &= \frac{1}{l^2\lambda(l)}\sum_{\substack{V\in\mathbb{N}\\ l\le V\le\sigma l^2}}\alpha^V D(V,l)\\ &= \frac{1}{l^2\lambda(l)}\sum_{\substack{n\in\mathbb{N}\cup\{0\}\\ n\le\sigma l^2-l}}\alpha^{n+l} D(n+l,l)\\ &=: G_l(\sigma)\to\int_0^\sigma p(x)\ dx,\quad \text{as } l\to\infty.\end{aligned}$$

Now

$$\frac{G_l(\sigma)}{F_l(\sigma)} = \frac{Z_l(\alpha)}{l^2\lambda(l)} \tag{5.36}$$

for every $l \geq 3$ and $\sigma \in [0,\infty]$. On the other hand, by fixing $\sigma = \infty$ we obtain

$$\frac{G_l(\infty)}{F_l(\infty)} = G_l(\infty) \to \int_0^\infty p(x)\ dx,\quad \text{as } l\to\infty.$$

Hence

$$\lim_{l\to\infty}\frac{Z_l(\alpha)}{l^2\lambda(l)} = \int_0^\infty p(x)\ dx$$

and by Equation 5.36 we may conclude that

$$\lim_{l\to\infty} F_l(\sigma) = \frac{\int_0^\sigma p(x)\ dx}{\int_0^\infty p(x)\ dx} \tag{5.37}$$

□

To be able to write the distribution of the limiting random variable explicitly we still need to determine an explicit value for the integral of the denominator of the expression above.

Lemma 5.19.

$$\int_0^\infty p(x)\ dx = \frac{81}{2048\sqrt{\pi}}$$

Proof. Integration by parts yields

$$\begin{aligned}\int_0^\infty x^{-\frac{3}{2}}e^{-\frac{1}{6x}}\ dx &= \Big|_0^\infty - 2x^{-\frac{1}{2}}e^{-\frac{1}{6x}} + 2\int_0^\infty x^{-\frac{1}{2}}\frac{1}{6}x^{-2}e^{-\frac{1}{6x}}\ dx\\ &= \frac{1}{3}\int_0^\infty x^{-\frac{5}{2}}e^{-\frac{1}{6x}}\ dx.\end{aligned}$$

By changing the integration variable from x to t, where $t^2 = \frac{1}{6x}$, we obtain

$$\begin{aligned}
\int_0^\infty x^{-\frac{3}{2}} e^{-\frac{1}{6x}} \, dx &= \frac{1}{3} \int_0^\infty \left(6t^2\right)^{\frac{3}{2}} e^{-t^2} t^{-3} \, dt \\
&= \frac{6^{\frac{3}{2}}}{3} \int_0^\infty e^{-t^2} \, dt \\
&= \sqrt{6\pi}.
\end{aligned}$$

Hence, by combining the two results above we may conclude that

$$\begin{aligned}
\int_0^\infty p(x) \, dx &= \frac{9\sqrt{\frac{3}{2}}}{2048\pi} \int_0^\infty x^{-\frac{5}{2}} e^{-\frac{1}{6x}} \, dx \\
&= \frac{81}{2048\sqrt{\pi}}
\end{aligned}$$

$\square$

By Equation 5.37 and Lemma 5.19, the random variables X_l converge in distribution to a random variable X, where the probability density function $f_X(x)$ of X satisfies

$$f_X(x) = \begin{cases} \dfrac{\sqrt{\frac{3}{2\pi}}}{9x^{\frac{5}{2}} e^{\frac{1}{6x}}}, & x > 0. \\ 0, & \text{otherwise.} \end{cases}$$

The probability density function of X is presented in Figure 5.10 with linear scale and in Figure 5.11 with logarithmic scale. The probability mass functions of X_5, X_{10} and X_{20} are presented in Figures 5.12, 5.13 and 5.14 respectively. The cumulative distribution function F_X of X is illustrated in Figure 5.15 and the cumulative distribution functions of X_5, X_{10} and X_{20} are illustrated in Figures 5.16, 5.17 and 5.18 respectively.

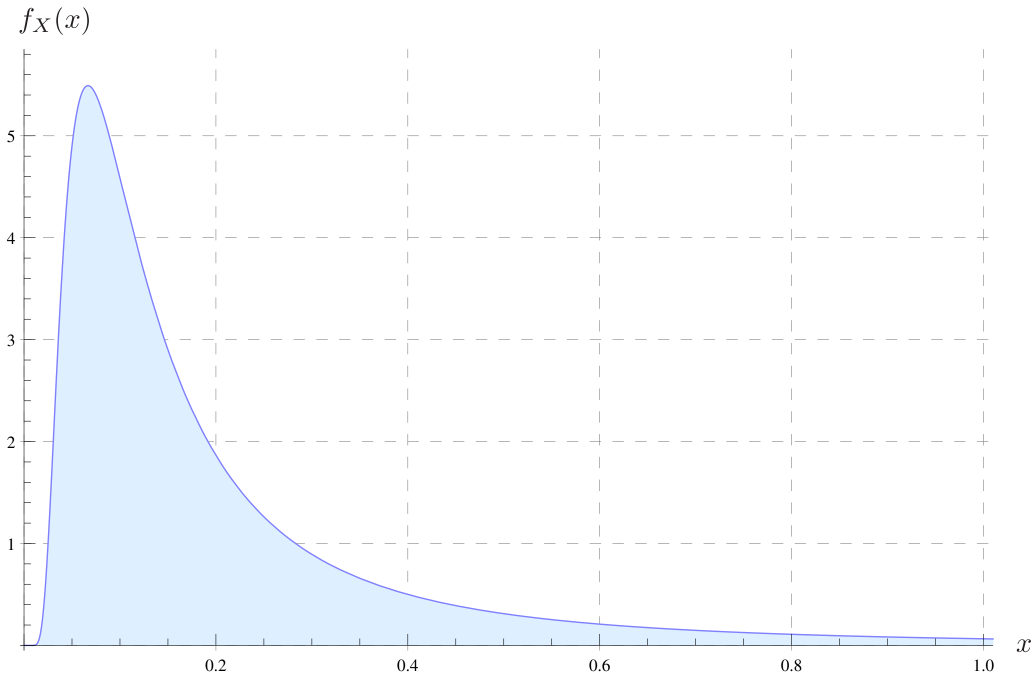


Figure 5.10: The probability density function of X, linear scale.

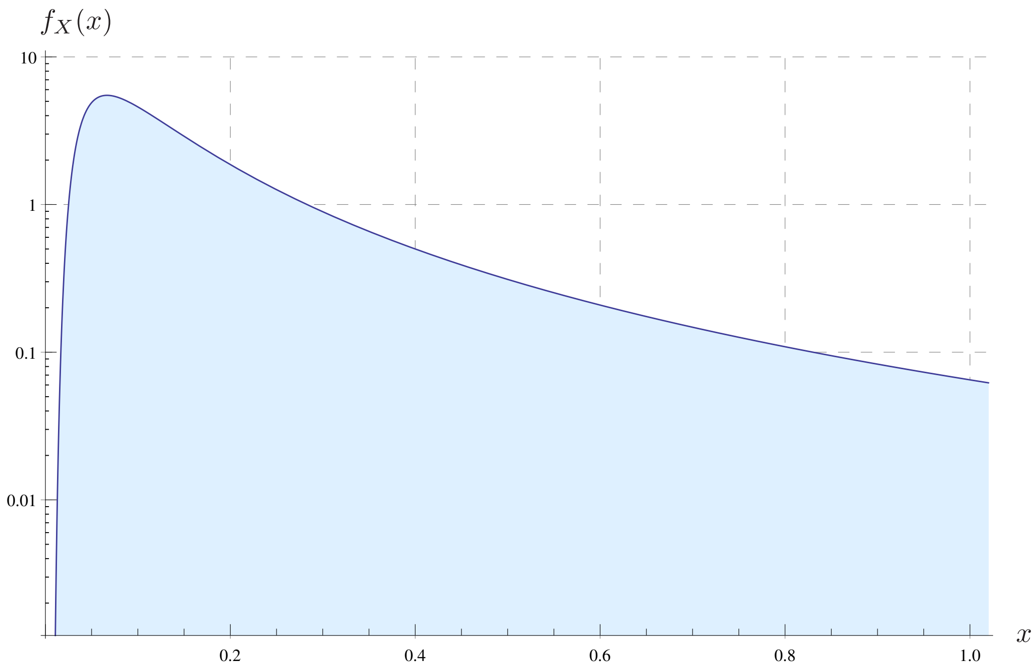


Figure 5.11: The probability density function of X, logarithmic scale.

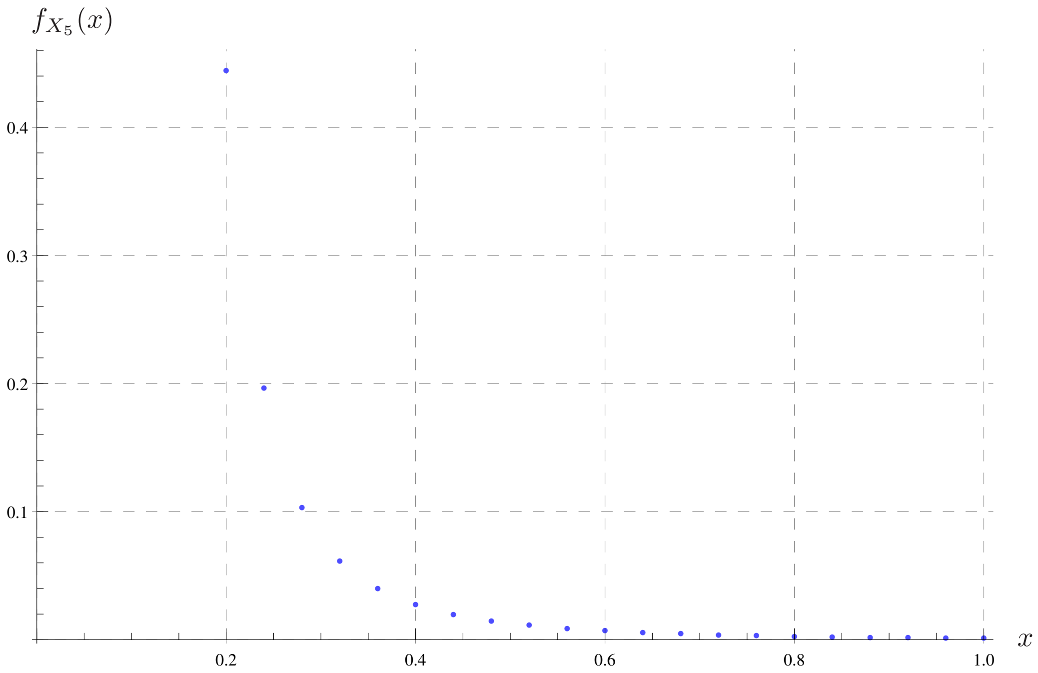


Figure 5.12: The probability mass function of X_5.

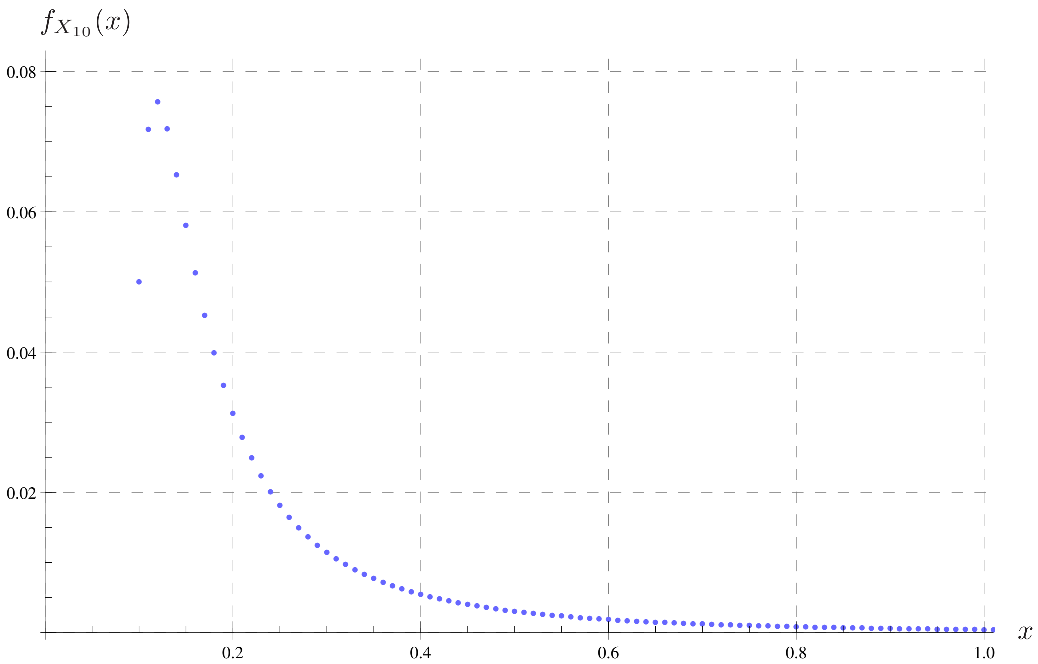


Figure 5.13: The probability mass function of X_{10}.

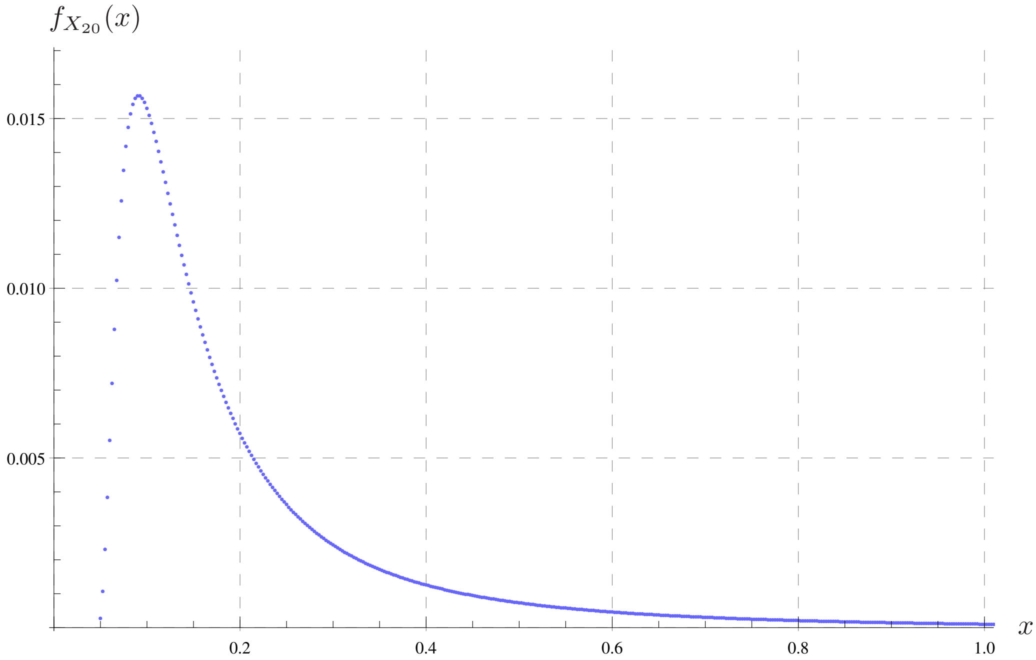


Figure 5.14: The probability mass function of X_{20}.

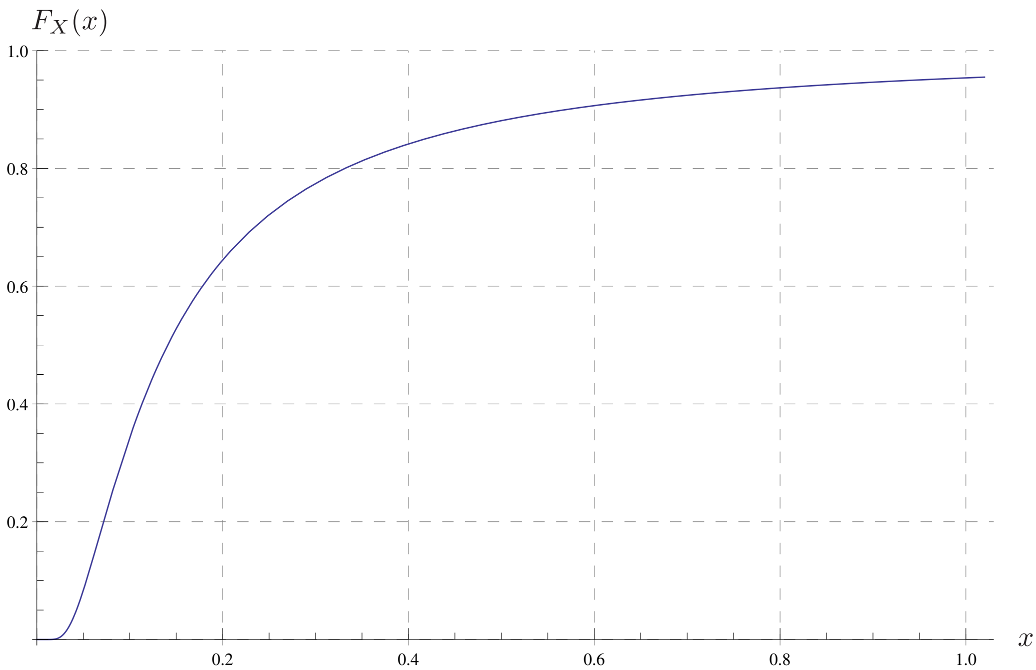


Figure 5.15: The cumulative distribution function of X.

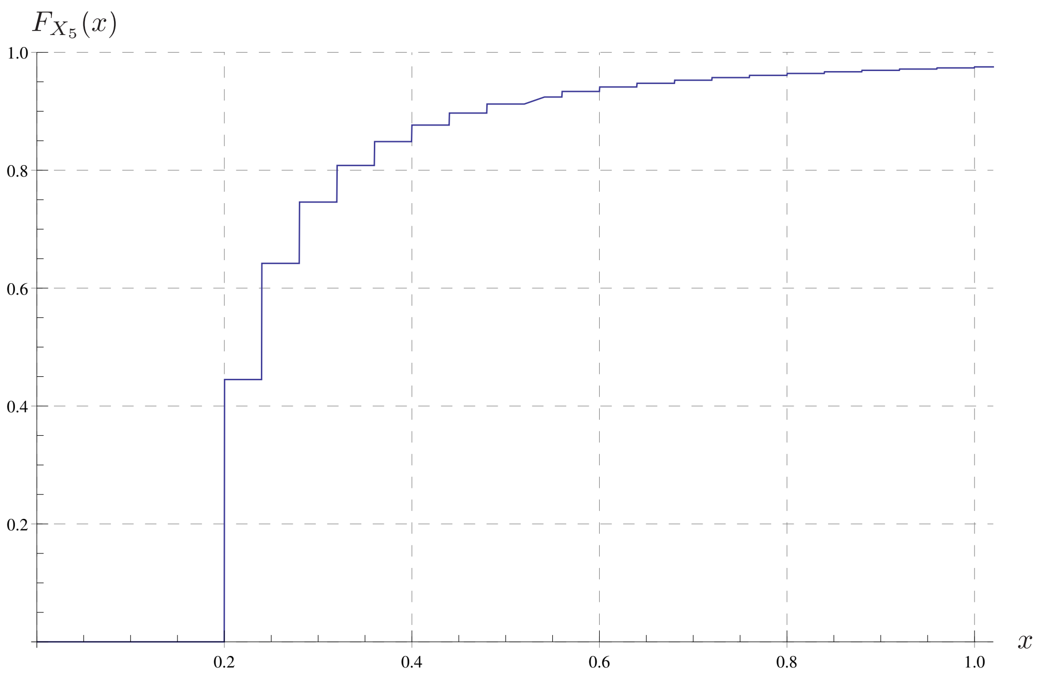


Figure 5.16: The cumulative distribution function of X_5.

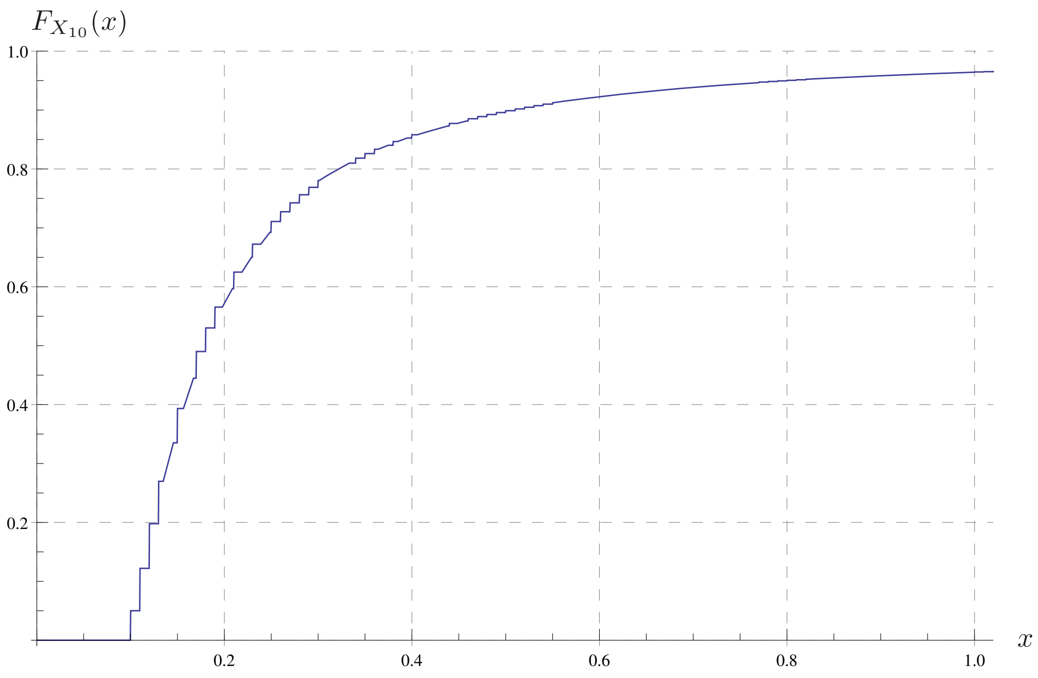


Figure 5.17: The cumulative distribution function of X_{10}.

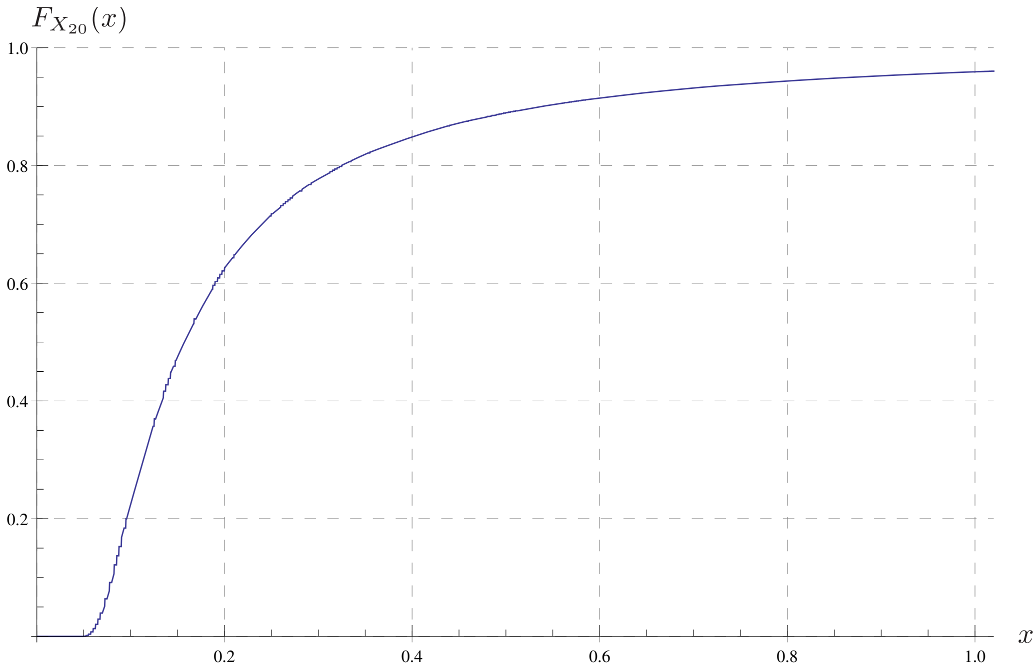


Figure 5.18: The cumulative distribution function of X_{20}.

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